Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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# Problems for Recitation 3

**False Theorem**: If every vertex in a vertex-colored simple graph is adjacent to a vertex of the same color, then the graph is monochromatic.

**False Proof**: By induction on the number of vertices in the graph. Let $P(n)$ be the predicate, "If every vertex in an $n$-vertex vertex-colored simple graph is adjacent to a vertex of the same color, then the graph is monochromatic."

In the base case, $P(1)$, the graph contains only one vertex and so is certainly monochromatic.

the inductive step, we assume $P(n)$ in order to prove $P(n+1)$. For $n \geq 1$, let $G = (V, E)$ be a vertex-colored graph with $n + 1$ vertices such that every vertex is adjacent to a vertex of the same color, and let $v \in V$ be an arbitrary vertex. Let $G'$ be the subgraph of $G$ obtained by deleting $v$. Since $G'$ is an $n$-vertex graph, we have by induction that $G'$ is monochromatic. By hypothesis, $v$ is adjacent to a vertex $v'$ of the same color. Since $v'$ must be a vertex of $G'$, it follows that $v$ has the same color as all the other vertices. Therefore, $G$ is indeed monochromatic. ■

**Problem 1.** Pinpoint the flaw in this proof.

**Problem 2.** How many simple graphs (up to isomorphism) are there with $|V| = |E| = 5$?

**Problem 3.** Let $A$ be the adjacency matrix for a digraph $G$ with $n$ vertices. Note that $A$ is a zero-one valued $n \times n$ matrix. In this problem we use the Boolean "OR" operation, $\vee$, instead of "+" when multiplying zero-one valued matrices (this is called "boolean" matrix product).

**(a)** For $k \in \mathbb{N}$, let $A^k$ be the $k$th *Boolean* power of $A$. Prove that ${A^k}_{ij} = 1$ iff there is a directed path of length $k$ from the $i$th to the $j$th vertices of $G$.

**(b)** Let $B = A \vee I$ where $I$ is the $n \times n$ identity matrix. Prove that ${B^k}_{ij} = 1$ iff there is a directed path of length $\leq k$ from the $i$th to the $j$th vertices of $G$.

**(c)** Conclude that ${B^n}_{ij} = 1$ iff there is a directed path from the $i$th to the $j$th vertex.

**Problem 4.** The *complement*, $\overline{G}$, of a simple graph, $G$, is defined to be the graph which has the same vertices as $G$ such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$.

**(a)** A subset, $A$, of the vertices of a simple graph, $G$, such that no two vertices in $A$ are adjacent in $G$, is called an *anti-clique* of $G$. Prove that $G$ contains an anti-clique of size $m$ iff $\overline{G}$ contains a copy of $K_m$.

**(b)** A simple graph, $G$, is called *self-complementary* iff $G$ and $\overline{G}$ are isomorphic. Prove that if $G$ is a self-complementary simple graph with $n$ vertices, then

$$n \equiv 0 \pmod{4} \text{ or } n \equiv 1 \pmod{4}.$$

*Hint:* How many edges altogether are there in $G$ and $\overline{G}$?

**(c)** For integer $n > 0$, define a simple graph $G_n$ as follows: the vertices are the integers $1, 2, \ldots, 2n$. Vertices $i \neq j$ are adjacent iff $i, j \leq n$ or ($i \leq n$ and $j > n$ and $i \equiv j \pmod{2}$).

Prove that $G_n$ is not $(n-1)$-colorable.

**(d)** Prove that $G_n$ is self-complementary.

**Problem 5.** Let $G$ be a simple graph with $n > 0$ vertices. Prove that if $G$ has no vertex of degree $\geq m$, then $G$ has an anti-clique of size $\geq n/m$.