Problems for Recitation 2

1. An n-player tournament, consists of some set of $n \geq 2$ elements called *players*, and, for every two distinct players, a specification that one of the players beats the other. That is, player p_1 beats player p_2 iff p_2 does not beat p_1, for all players $p_1 \neq p_2$, and no player beats him/herself.

   A sequence of players $p_1, p_2, \ldots, p_k$ (for some $k \leq n$), such that player p_i beats player p_{i+1} for $1 \leq i < k$ is called a *ranking* of these players. If also player p_k beats player p_1, the ranking is called a k-*cycle*.

   (a) Prove that in every tournament, either there is a "champion" player that beats every other player, or there is a 3-cycle.

   (b) A *consistent ranking* is a sequence $p_1, p_2, \ldots, p_n$ of all n players in the tournament such that p_i beats p_j iff $i < j$, for $1 \leq i, j \leq n$. Conclude that a tournament has no consistent ranking iff some subset of three of its players has no consistent ranking.

2. The well-matched strings of left and right parentheses, *WMP*'s for short, are defined inductively as follows:

   - The string of length two consisting of a left parenthesis followed by a right parenthesis, namely, `()`, is a WMP.
   - If $W_1 W_2 \ldots W_k$ is a sequence of $k \geq 1$ WMP's, then `(`$W_1 W_2 \ldots W_k$`)` is a WMP.

   The *count* of a string of one or more parentheses is defined as follows:

   - count(`)`) $= -1$,
   - count(`(`) $= 1$,
   - of s is a string of parentheses and p is a single parenthesis, then $\text{count}(sp) = \text{count}(s) + \text{count}(p)$.

   A string, Z, of one or more parentheses is a *well-counting* string of parentheses, a *WCP* for short, iff $\text{count}(Z) = 0$ and $\text{count}(X) > 0$ for any proper prefix, X, of Z. (X is a *prefix* of Z means that $Z = XY$ for some string Y. X is a *proper* prefix if it is a prefix of Z and its length is positive and strictly less than the length of Z.)

   (a) Prove that every WMP is a WCP.

   (b) Conversely, prove that every WCP is a WMP.