Massachusetts Institute of Technology Handout 55
6.042J/18.062J: Mathematics for Computer Science May 5, 1998 (typos corrected 9:30pm)
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Problem Set 10 Solutions

Problem 1. [**8 points**]

Show that if two positive random variables R and T are independent, then $2R$ and e^T are also independent.

Solution

$$\begin{aligned}\Pr((2R = r) \wedge (e^T = t)) &= \Pr((R = r/2) \wedge (T = \ln t)) \\ &= \Pr(R = r/2) Pr(T = \ln t) \\ &= \Pr(2R = r) Pr(e^T = t)\end{aligned}$$

Problem 2. [**12 points**]

(a) [**6 points**]

Let R_1 and R_2 be two independent random variables. R_1 takes the values $0, 1$ with equal probability. R_2 has the following distribution.

$\Pr(R_2 = 1) = 1/2$
$\Pr(R_2 = 2) = 1/4$
$\Pr(R_2 = 4) = 1/4$

Compute the distribution of $2^{R_1} R_2$.

Solution

Let $X = 2^{R_1} R_2$.

$\Pr(X = 1) = \Pr(R_1 = 0 \wedge R_2 = 1) = 1/4$.
$\Pr(X = 2) = \Pr((R_1 = 0 \wedge R_2 = 2) \vee (R_1 = 1 \wedge R_2 = 1) = 3/8$.
$\Pr(X = 4) = \Pr((R_1 = 0 \wedge R_2 = 4) \vee (R_1 \wedge R_2 = 2)) = 1/4$.
$\Pr(X = 8) = \Pr(R_1 = 1 \wedge R_2 = 4) = 1/8$.

(b) [**6 points**]

Now suppose that R_1 and R_2 are no longer independent. In particular, $R_1 = 0$ if R_2 is 1, and $R_1 = 1$ when R_2 is 2 or 4. R_2 has the same distribution as in part (a).

Compute the distribution of $2^{R_1} R_2$.

Solution

$\Pr(X = 1) = \Pr(R_2 = 1) = 1/2$.
$\Pr(X = 4) = \Pr(R_2 = 2) = 1/4$.
$\Pr(X = 8) = \Pr(R_2 = 4) = 1/4$.

Problem 3. **[24 points]**

Consider the process of tossing a biased coin with $\text{Prob}(\text{Heads}) = p$. Assume that all tosses are mutually independent.

(a) **[5 points]**

What is the probability that exactly k of the first 100 tosses will turn up heads?

Solution

The result of each toss is the outcome of a Bernoulli trial with probability of success p. Therefore the number of heads observed in the first 100 tosses is binomially distributed.

$$\Pr(k) = \binom{100}{k} p^k (1-p)^{100-k}$$

(b) **[7 points]**

Consider the first 100 tosses. For what values of k is the probability of getting exactly k heads greater than the probability of getting exactly $k-1$ heads? For what values of k is it less? What value of k is the most likely number of heads?

Solution

Consider the ratio of the probability of k heads over the probability of $k-1$ heads.

$$\begin{aligned}
\frac{\binom{100}{k} p^k (1-p)^{100-k}}{\binom{100}{k-1} p^{k-1} (1-p)^{100-k+1}} &= \frac{\frac{100!}{k!(100-k)!} p}{\frac{100!}{(k-1)!(100-k+1)!}(1-p)} \\
&= \frac{(100-k+1)p}{k(1-p)}
\end{aligned}$$

This quantity is greater or equal to 1 if and only if $(100-k+1)p \geq k(1-p) \iff 101p \geq k$. So, for $k \leq 101p$ the probability increases from $k-1$ to k, and for $k \geq 101p$, the probability decreases. The value of k that maximizes the probability is thus $\lfloor 101p \rfloor$.

(c) [**6 points**]

Define W_k to be the waiting time until the kth head, that is, the number of tosses necessary to reach a total of k heads. For example, if the sequence of flips is $HTTHHT\ldots$, then $W_1 = 1, W_2 = 4$, and $W_3 = 5$.

Determine $E(W_k)$ in terms of k and p. (Hint: determine $E(W_1)$ and use linearity of expectation.)

Solution

$E(W_1)$ is the mean time to failure. From class, we have that $E(W_1) = 1/p$. Since future heads are independent of past ones, the time to get heads twice is $W_1 + W_1 = 2W_1$, and more generally the time to see k heads is kW_1. Therefore

$$\begin{aligned} E[W_k] &= E[kW_1] \\ &= kE[W_1] \\ &= k/p \end{aligned}$$

(d) [**6 points**]

Determine $\text{Prob}(W_k = n)$ in terms of k, p, and n.

Solution

Consider the event $\{W_k = n\}$. This means that the nth flip was heads, and there were $k-1$ heads on the previous $n-1$ flips.

Since the arrivals at each second are independent, we have:

$$\begin{aligned} \Pr(W_k = n) &= \Pr(k-1 \text{ heads in } n-1 \text{ flips}) P(1 \text{ arrival on the } n\text{th second}) \\ &= \binom{n-1}{k-1} p^{k-1}(1-p)^{l-k} p \\ &= \binom{n-1}{k-1} p^{k}(1-p)^{l-k} \end{aligned}$$

Problem 4. [**15 points**]

Chuck-A-Luck is played in many casinos. One version of it is as follows. There is a cage with three dice in it. The player pays \$1 to guess a number between 1 and 6. The three dice are then rolled. If one die shows the player's number, he wins \$2 (i.e., he wins his old dollar back plus a dollar profit). If two dice show his number, he wins \$3. If three dice show his number, he wins \$4. If no die shows the number, he wins nothing.

If Bobo spends his last dollar on a game of Chuck-A-Luck, what is his expected wealth after the game?

Solution

Let random variable Y be the number of dice showing the chosen number, and X be his wealth after one play. Then $X = 0$ if $Y = 0$ and $X = i+1$, if $Y = i$, for $i = 1, 2, 3$. $P(Y = i) = \binom{3}{i}\left(\frac{1}{6}\right)^i\left(\frac{5}{6}\right)^{3-i}$, $i = 0, 1, 2, 3$. Therefore,

$$\begin{aligned}
E(X) &= 0 \cdot P(Y = 0) + \sum_{i=1}^{3}(i+1) \cdot P(Y = i) \\
&= \sum_{i=1}^{3}(i+1) \cdot \binom{3}{i}\left(\frac{1}{6}\right)^i\left(\frac{5}{6}\right)^{3-i} \\
&= \frac{1}{216}(2 \cdot 75 + 3 \cdot 15 + 4 \cdot 1) \\
&= 199/216 = .921
\end{aligned}$$

Problem 5. [**14 points**]

Consider a system consisting of 2 components, each of which can fail at each step with probability p. Failures at different steps and/or components are mutually independent (in particular, this means that failure need not be permanent). The steps are numbered $1, 2, \ldots$.

(a) [**7 points**]

Assume that the system fails iff at least one component fails. Find the system's mean time to failure.

Solution

At any particular step, the probability that at least one component fails is $1 - (1-p)^2 = 2p - p^2$.

Let X be the random variable whose value is the time to failure. The distribution of X is

$\Pr(X = i) = (1-p)^{2(i-1)}(2p - p^2)$.

From class, the mean time to failure $E(X) = \frac{1}{2p-p^2}$.

(b) [**7 points**]

Now assume that the system fails iff both components fail simultaneously. Find the system's mean time to failure.

Solution

The probability that both components will fail simultaneously is p^2.

Let Y be the random variable whose value is the step number of the first step at which both components fail simultaneously. The distribution of Y is

$\Pr(Y = i) = (1 - p^2)^{i-1}p^2$.
so the mean time to failure is $E(Y) = \frac{1}{p^2}$.

Problem 6. **[15 points]**

Suppose that we toss two hundred mutually independent coins, each with $\Pr(\text{Heads}) = .4$. Use the formula given in lecture to estimate the probability that 60 or fewer coins will be heads.

Solution

We note that $n = 200$, $p = .4$, and $\alpha = 60/200 = .3$.

$$\begin{aligned}
F_{200,.4}(60) &\leq \frac{1 - .3}{1 - \frac{.3}{.4}} f_{200,.4}(60) \\
&= \frac{.7}{.25} \frac{2^{\left(.3\log_2\left(\frac{.4}{.3}\right) + (1-.3)\log_2\left(\frac{1-.4}{1-.3}\right)\right)\times 200} e^{a_{200} - a_{60} - a_{140}}}{\sqrt{2\pi(.3)(1-.3)(200)}} \\
&\approx (2.8)\frac{2^{(.1245-.1557)\times 200} \times .99...}{16.24} \\
&\approx 2.2811 \times 10^{-3}
\end{aligned}$$

So $\Pr(\text{less than or equal to 60 heads}) \leq 2.2811 \times 10^{-3}$.

Problem 7. **[12 points]**

An airport bus deposits 8 passengers at 6 stops. Assume that each passenger is as likely to get off at any stop as another and that they act independently. The bus stops only if someone wants to get off. What is the probability that nobody gets off at the third stop? What is the expected number of stops it will make?

Solution

For each individual, the probability he *does not* get off at the third stop is $\frac{5}{6}$. Therefore, the probability that no one gets off at the third stop is $(\frac{5}{6})^8 = 0.2326$.

Let I_k be an indicator variable for whether the bus stops at the k-th stop. Then, $X = \sum_k I_k$ is the total number of stops. By linearity of expectation,

$$E(X) = E(I_1) + E(I_2) + E(I_3) + E(I_4) + E(I_5) + E(I_6)$$

For each I_k, the $E(I_k)$ is the probability the bus stops at the k-th stop, or $1 - 0.2326 = 0.7674$. Therefore, $E(X) = 6(0.7674) = 4.6044$.