Problem Set 10

Due: May 5, 1998.

Reading assignment: 6.042 Fall 1997 Lecture Notes 21 and 22

Problem 1. [**8 points**]

Show that if two positive random variables R and T are independent, then $2R$ and e^T are also independent.

Problem 2. [**12 points**]

(a) [**6 points**] Let R_1 and R_2 be two independent random variables. R_1 takes the values $0, 1$ with equal probability. R_2 has the following distribution.

$\Pr(R_2 = 1) = 1/2$
$\Pr(R_2 = 2) = 1/4$
$\Pr(R_2 = 4) = 1/4$

Compute the distribution of $2^{R_1}R_2$.

(b) [**6 points**] Now suppose that R_1 and R_2 are no longer independent. In particular, $R_1 = 0$ if R_2 is 1, and $R_1 = 1$ when R_2 is 2 or 4. R_2 has the same distribution as in part (a).

Compute the distribution of $2^{R_1}R_2$.

Problem 3. [**24 points**]

Consider the process of tossing a biased coin with Prob(Heads) $= p$. Assume that all tosses are mutually independent.

(a) [**6 points**] What is the probability that exactly k of the first 100 tosses will turn up heads?

(b) [**6 points**] Consider the first 100 tosses. For what values of k is the probability of getting exactly k heads greater than the probability of getting exactly $k-1$ heads? For what values of k is it less? What value of k is the most likely number of heads?

(c) [**6 points**] Define W_k to be the waiting time until the kth head, that is, the number of tosses necessary to reach a total of k heads. For example, if the sequence of flips is $HTTHHT\ldots$, then $W_1 = 1, W_2 = 4$, and $W_3 = 5$. Determine $E(W_k)$ in terms of k and p. (Hint: determine $E(W_1)$ and use linearity of expectation.)

(d) [**6 points**] Determine $\text{Prob}(W_k = n)$ in terms of k, p, and n.

Problem 4. [**15 points**] Chuck-A-Luck is played in many casinos. One version of it is as follows. There is a cage with three dice in it. The player pays \$1 to guess a number between 1 and 6. The three dice are then rolled. If one die shows the player's number, he wins \$2 (i.e., he wins his old dollar back plus a dollar profit). If two dice show his number, he wins \$3. If three dice show his number, he wins \$4. If no die shows the number, he wins nothing.

If Bobo spends his last dollar on a game of Chuck-A-Luck, what is his total expected wealth after the game?

Problem 5. [**14 points**] Consider a system consisting of 2 components, each of which can fail at each step with probability p. Failures at different steps and/or components are mutually independent (in particular, this means that failure need not be permanent). The steps are numbered $1, 2, \ldots$.

(a) [**7 points**] Let X be the random variable whose value is the step number of the first step at which there is a failure (in either component). Find $E[X]$.

(b) [**7 points**] Let Y be the random variable whose value is the step number of the first step at which both components fail simultaneously. Find $E[Y]$.

Problem 6. [**15 points**] Suppose that we toss two hundred mutually independent coins, each with $\Pr(\text{Heads}) = 0.4$

Use the formula given in lecture to estimate the probability that 60 or fewer coins will be heads.

Problem 7. [**12 points**] An airport bus deposits 8 passengers at 6 stops. Assume that each passenger is as likely to get off at any stop as another and that they act independently. The bus stops only if someone wants to get off. What is the probability that nobody gets off at the third stop? What is the expected number of stops it will make?