Problem Set 9

Due: Tuesday, April 28.

Reading assignment: Rosen §4.4–5; 6.042 Fall 97, Notes for Lectures 18-20

Problem 1. [15 pts]

Suppose the game *Let's Make A Deal* is changed slightly. Instead of having 3 doors with 1 grand prize, in the new game there are 4 doors with 2 grand prizes. The rule of the game remains the same. Therefore the game now becomes:

> At one stage of the game, a contestant is shown 4 doors. The contestant knows that there are 2 grand prizes behind 2 of the doors and that there are goats behind the 2 other doors. At the beginning, the contestant picks a door. To build suspense, the assistant of the game then opens a *different* door, revealing a goat. The contestant can then stick with his original door or switch to another unopened door. He wins a prize only if he now picks the door with the prize.

In the new game, what is the probability of winning with the "stick" strategy? How about the "switch" strategy? Can you explain why the two probabilities do not sum up to 1?

Problem 2. [20 pts]

A sequence of numbers $< 0, 2, 2, 2, 2, 5, 5, 7, 7 >$ is randomly permuted to form a 9-digit integer, in such a way that the digit 0 does not come first. (In other words, if the digit 0 comes first in the permutation, we try a new permutation until this does not occur.) Let p be the 9-digit number. Find the probabilities of the following:

(a) p is a multiple of 2.

(b) p is a multiple of 25.

(c) the first and the last digits of p are 2.

(d) p is symmetric. (i.e., the number does not change if the sequence of digits is reversed)

Problem 3. [10 pts]

In a lottery, 6 distinct integers are drawn randomly from 1 to 45 without replacement. What is the probability that the maximum of the 6 numbers is k, where $1 \leq k \leq 45$?

Problem 4. In a complex system with 1,000 different components, each component is known to fail with probability at most $\frac{1}{10{,}000}$. The failures may not be independent events.

(a) Prove that $\frac{1}{10}$ is an upper bound for the probability that one or more components in the system fail. [10 pts]

(b) Show that this upper bound is optimal by giving a case where the probability that one or more components in the system fail is exactly $\frac{1}{10}$. [5 pts]

Problem 5. There are 4 coins in a box, namely c_1, c_2, c_3, c_4. The probability p_i that the coin c_i gives a Head is $\frac{1}{i}$ for $1 \leq i \leq 4$. A coin is selected from the box randomly, and is being tossed until a Head is seen.

(a) Write down the sample space of the experiment. [2 pts]

(b) What is the probability that a Head is first seen in the 2^{nd} toss? [9 pts]

(c) Given that a Head is first seen in the 2^{nd} toss, what is the probability that c_i is selected from the box? [9 pts]

Problem 6. [10 pts]

A, B, C are events of an experiment. If C is independent of A and C is independent of B, is C independent of $A \cap B$? Prove your answer or give a counterexample.

Problem 7. [10 pts]

A house is installed with a fire alarm to prevent fire accidents. If there is a fire, the probability for the fire alarm being turned on is 96%. Even without a fire, the alarm may be turned on with a probability of 2%. Suppose the probability for a fire accident to happen in the house is 1%. If the fire alarm is now on, what is the conditional probability that there is really a fire accident?