Problem Set 8 Solutions

Problem 1. First buy one donut of each flavor. Then the problem is reduced to choosing eight donuts without constraints. This is an 8-combination with repetition from a 4-element set, so the answer is

$$\binom{4+8-1}{8} = \binom{11}{8} = 165.$$

Alternatively, we can use the formula given in lecture for an r-combination with at least one of each item. Then we have

$$\binom{12-1}{4-1} = \binom{11}{3} = 165.$$

Problem 2. The string is eleven letters long, and there are five A's, two B's, two R's, one C, and one D. The multinomial coefficient is

$$\binom{11}{5,2,2,1,1} = \frac{11!}{5!2!2!1!1!} = 83,160.$$

Problem 3. **(a)** $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{(1-x)}.$

(b) $1 + x + x^2 + x^3.$

(c) $\displaystyle\sum_{n=2}^{\infty} x^n = x^2 \sum_{n=0}^{\infty} x^n = \frac{x^2}{(1-x)}.$

(d) $\displaystyle\sum_{n=4}^{\infty} x^n = x^4 \sum_{n=0}^{\infty} x^n = \frac{x^4}{(1-x)}.$

(e) $\displaystyle x^5 + \sum_{n=3}^{\infty} x^n = x^5 + x^3 \sum_{n=0}^{\infty} x^n = x^5 + \frac{x^3}{(1-x)}.$

(f) $\displaystyle\sum_{n=0}^{\infty} x^{4n} = \sum_{n=0}^{\infty} (x^4)^n = \frac{1}{(1-x^4)}.$

(g) $\sum_{n=0}^{\infty}(n+1)^2x^n = \frac{1}{x}\sum_{n=0}^{\infty}(n+1)^2x^{n+1} = \frac{1}{x}\sum_{n=1}^{\infty}n^2x^n = \frac{1}{x}\sum_{n=0}^{\infty}n^2x^n = \frac{1}{x}\frac{x(1+x)}{(1-x)^3} = \frac{1+x}{(1-x)^3}$.

(h) $\sum_{n=0}^{\infty}(n+1)(n+2)x^n = \frac{1}{x}\sum_{n=0}^{\infty}(n+1)(n+2)x^{n+1} = \frac{1}{x}\sum_{n=1}^{\infty}n(n+1)x^n = \frac{1}{x}\sum_{n=1}^{\infty}n^2x^n + \frac{1}{x}\sum_{n=1}^{\infty}nx^n = \frac{1}{x}\sum_{n=0}^{\infty}n^2x^n + \frac{1}{x}\sum_{n=0}^{\infty}nx^n = \frac{1}{x}\frac{x(1+x)}{(1-x)^3} + \frac{1}{x}\frac{x}{(1-x)^2} = \frac{1+x}{(1-x)^3} + \frac{1}{(1-x)^2} = \frac{2}{(1-x)^3}$.

(i) $\sum_{n=0}^{\infty}n^2(n+2)x^n = \sum_{n=0}^{\infty}n^3x^n + 2\sum_{n=0}^{\infty}n^2x^n = \frac{x+4x^2+x^3}{(1-x)^4} + 2\frac{x+x^2}{(1-x)^3} = \frac{3x+4x^2-x^3}{(1-x)^4}$.

(j) $\sum_{n=0}^{\infty}(3^n+5^n+7^n)x^n = \sum_{n=0}^{\infty}3^nx^n + \sum_{n=0}^{\infty}5^nx^n + \sum_{n=0}^{\infty}7^nx^n = \sum_{n=0}^{\infty}(3x)^n + \sum_{n=0}^{\infty}(5x)^n + \sum_{n=0}^{\infty}(7x)^n = \frac{1}{1-3x} + \frac{1}{1-5x} + \frac{1}{1-7x}$.

(k) $\sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x$. (Taylor Series)

(l) (Use Convolution)

Let $(\sum_{k=0}^{\infty}a_kx^k)(\sum_{k=0}^{\infty}b_kx^k) = \sum_{k=0}^{\infty}c_kx^k$ then, $c_k = \sum_{i=0}^{k}a_ib_{k-i}$

If $b_i = 1$ for all i, $c_k = \sum_{i=0}^{k}a_i$

Therefore, $\sum_{k=0}^{\infty}(\sum_{i=0}^{k}\frac{1}{i!})x^k = (\sum_{k=0}^{\infty}\frac{1}{k!}x^k)(\sum_{k=0}^{\infty}x^k) = (e^x)(\frac{1}{1-x}) = \frac{e^x}{1-x}$.

Problem 4. **(a)** $\frac{8x}{4x^2-1} = \frac{8x}{(2x-1)(2x+1)} = \frac{2}{2x-1} + \frac{2}{2x+1}$.

(b) $x^2-4x+4+\frac{2x-1}{(x+2)(x+1)} = x^2-4x+4+\frac{5}{x+2}-\frac{3}{x+1}$.

(c) $\frac{2}{2x-1} + \frac{2}{2x+1} = -\frac{2}{1-2x} + \frac{2}{1+2x} = -2\sum_{n=0}^{\infty}(2x)^n + 2\sum_{n=0}^{\infty}(-2x)^n$.

Therefore, the 17th coefficient(coefficient of x^{16}) is $-2(2)^{16}+2(-2)^{16} = 0$.

(or $-2(2)^{17} + 2(-2)^{17} = -2^{18} - 2^{18} = -2^{19}$ if you calculate the coefficient of x^{17}).

(Alternative solution)

$$\frac{8x}{4x^2-1} = \frac{-8x}{1-4x^2} = -8x\sum_{n=0}^{\infty}(4x^2)^n = -8x\sum_{n=0}^{\infty}4^n x^{2n} = \sum_{n=0}^{\infty} -2^{2n+3}x^{2n+1}.$$

Therefore, the 17th coefficient(coefficient of x^{16}) is 0. (or -2^{19} if you calculate the coefficient of x^{17}).

(d) $x^2 - 4x + 4 + \dfrac{5}{x+2} - \dfrac{3}{x+1} = x^2 - 4x + 4 + \dfrac{5}{2}\sum_{n=0}^{\infty}(-\frac{1}{2}x)^n - 3\sum_{n=0}^{\infty}(-x)^n$

Therefore,

$g_0 = 4 + (\frac{5}{2})(-\frac{1}{2})^0 - 3(-1)^0 = \frac{7}{2}$

$g_1 = -4 + (\frac{5}{2})(-\frac{1}{2})^1 - 3(-1)^1 = -\frac{9}{4}$

$g_2 = 1 + (\frac{5}{2})(-\frac{1}{2})^2 - 3(-1)^2 = -\frac{11}{8}$

$g_3 = (\frac{5}{2})(-\frac{1}{2})^3 - 3(-1)^3 = \frac{43}{16}$

Problem 5. Let $A(x) = \sum_{i=0}^{\infty} a_i x^i$. Then, we get from the recurrence that

$$\begin{aligned}
A(x) &= 1 + 2x + 4x^2 + \sum_{i=3}^{\infty} a_i x^i \\
&= 1 + 2x + 4x^2 + \sum_{i=3}^{\infty}(2a_{i-2} + a_{i-3})x^i \\
&= 1 + 2x + 4x^2 + \sum_{i=3}^{\infty} 2a_{i-2}x^i + \sum_{i=3}^{\infty} a_{i-3}x^i \\
&= 1 + 2x + 4x^2 + 2x^2\sum_{i=1}^{\infty} a_i x^i + x^3\sum_{i=0}^{\infty} a_i x^i \\
&= 1 + 2x + 2x^2 + 2x^2\sum_{i=0}^{\infty} a_i x^i + x^3\sum_{i=0}^{\infty} a_i x^i \\
&= 1 + 2x + 2x^2 + 2x^2 A(x) + x^3 A(x)
\end{aligned}$$

since $a_0 = 2^0 = 1,\ a_1 = 2^1 = 2,\ a_2 = 2^2 = 4$. Solving for $A(x)$ we get

$$A(x) = \frac{2x^2+2x+1}{-x^3-2x^2+1} = \frac{-2x^2-2x-1}{(x+1)(x^2+x-1)}$$

Roots of x^2+x-1 are $\alpha_+ = \dfrac{-1+\sqrt{5}}{2}$ and $\alpha_- = \dfrac{-1-\sqrt{5}}{2}$.

Hence, we have:

$$A(x) = \frac{1}{1+x}\cdot\frac{-1}{(-1-\alpha_+)(-1-\alpha_-)}+\frac{1}{x-\alpha_+}\cdot\frac{-2\alpha_+^2-2\alpha_+-1}{(\alpha_++1)(\alpha_+-\alpha_-)}+\frac{1}{x-\alpha_-}\cdot\frac{-2\alpha_-^2-2\alpha_--1}{(\alpha_-+1)(\alpha_--\alpha_+)}$$

Some identities of α_+ and α_-:
$1+\alpha_+ = -\alpha_-,\ 1+\alpha_- = -\alpha_+,\ \alpha_+-\alpha_- = \sqrt{5},$
$\alpha_*^2+\alpha_*-1=0,$ where $*$ stands for either $+$ or $-$, $\alpha_+\alpha_- = -1.$
Employing these identities, we can write:

$$\begin{aligned}
A(x) &= \frac{1}{1+x}(\frac{-1}{\alpha_+\alpha_-})+\frac{1}{x-\alpha_+}(\frac{-3}{(-\alpha_-)\sqrt{5}})+\frac{1}{x-\alpha_-}(\frac{-3}{(-\alpha_+)(-\sqrt{5})})\\
&= \frac{1}{1+x}+\frac{3}{(\alpha_-)\sqrt{5}}\cdot\frac{1}{x-\alpha_+}+\frac{-3}{(\alpha_+)\sqrt{5}}\cdot\frac{1}{x-\alpha_-}\\
&= \frac{1}{1+x}+\frac{3}{\sqrt{5}}\cdot\frac{1}{1+\alpha_-x}-\frac{3}{\sqrt{5}}\cdot\frac{1}{1+\alpha_+x}
\end{aligned}$$

This implies that

$$\begin{aligned}
a_n &= (-1)^n+\frac{3}{\sqrt{5}}(-\alpha_-)^n-\frac{3}{\sqrt{5}}(-\alpha_+)^n\\
&= (-1)^n+\frac{3}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)\\
&= (-1)^n+3F(n) \text{ where } F(n) \text{ is the } n^{th} \text{ Fibonacci number}
\end{aligned}$$

We now verify by induction that indeed $a_n = (-1)^n+3F(n)$.

Let $P(n)$ be the statement that "For $k \le n,\ a_n = (-1)^n+3F(n)$". Then we have:

Proof.
Base cases:
$P(0): a_0 = 1+3F(0) = 1+0 = 1$
$P(1): a_1 = -1+3F(1) = -1+3 = 2$
$P(2): a_2 = 1+3F(2) = 1+3 = 4$

Inductive step. Assume $P(n)$ is true. Then

$$\begin{aligned}
a_{n+1} &= 2a_{n-1} + a_{n-2} \quad \text{by definition.} \\
&= 2\left((-1)^{n-1} + 3F(n-1)\right) + (-1)^{n-2} + 3F(n-2) \quad \text{by inductive hypothesis} \\
&= (-2+1)(-1)^{n-2} + 3\left(2F(n-1) + F(n-2)\right) \\
&= -(-1)^{n-2} + 3\left(F(n-1) + F(n-1) + F(n-2)\right) \\
&= (-1)^{n-1} + 3\left(F(n-1) + F(n)\right) \\
&= (-1)^{n+1} + 3F(n+1)
\end{aligned}$$

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Problem 6. **(a)** OGF for $\langle a_k \rangle = \frac{1}{1-x^2} \implies A(x) = (\frac{1}{1-x^2})^5$.

(b) OGF for $\langle a_k \rangle = \frac{1}{1-x^5} \implies A(x) = (\frac{1}{1-x^5})^5$.

(c) OGF for: $\langle a_1 \rangle = 1$, $\langle a_2 \rangle, \langle a_3 \rangle = 1 + x$, $\langle a_4 \rangle, \langle a_5 \rangle = \frac{1}{1-x} \implies A(x) = \frac{(1+x)^2}{(1-x)^2}$.

(d) OGF for: $\langle a_1 \rangle = x^2 + x^5 + x^8$, $\langle a_2 \rangle = x^4 + x^7 + x^{10}$, $\langle a_3 \rangle, \langle a_4 \rangle, \langle a_5 \rangle = \frac{1}{1-x} \implies A(x) = \frac{(x^2+x^5+x^8)(x^4+x^7+x^{10})}{(1-x)^3}$.

(e) OGF for $\langle a_k \rangle = \frac{x^{15}}{1-x} \implies A(x) = (\frac{x^{15}}{1-x})^5$.