Problem Set 8

Due: April 14, 1998.

Reading assignment: Rosen §4.6; Rosen Appendix 3; 6.042 Fall 97, Notes for Lecture 17.

Problem 1. [**11 points**] A donut shop carries just four flavors of donuts: glazed, chocolate, Boston creme, and strawberry. How many ways can we choose a box of 12 donuts so that we have at least one of each flavor?

Problem 2. [**10 points**] How many eleven-letter strings can be formed by permuting the letters in ABRACADABRA?

Problem 3. [**24 points**]

Compute the (ordinary) generating function for each of the following sequences.

(a) $\langle 1, 1, 1, 1, 1, \ldots \rangle$

(b) $\langle 1, 1, 1, 1, 0, 0, 0, 0, 0, \ldots \rangle$

(c) $\langle 0, 0, 1, 1, 1, 1, 1, \ldots \rangle$

(d) $\langle 0, 0, 0, 0, 1, 1, 1, 1, 1, \ldots \rangle$

(e) $\langle 0, 0, 0, 1, 1, 2, 1, 1, 1, 1, 1 \ldots \rangle$

(f) $\langle 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, \ldots \rangle$

(g) $\langle 1, 4, 9, 16, 25, \ldots \rangle$

(h) $\langle 2, 6, 12, 20, 30, 42, 56, \ldots \rangle$

(i) $\langle a_k \rangle = \left\langle k^2(k+2) \right\rangle$

(j) $\langle a_k \rangle = \left\langle 3^k + 5^k + 7^k \right\rangle$

(k) $\langle a_k \rangle = \langle 1/k! \rangle$

(l) $\langle a_k \rangle = \left\langle \sum_{i=0}^{k} \frac{1}{i!} \right\rangle$

Problem 4. [**15 points**] Find partial fraction decompositions for the following functions.

(a) $\dfrac{8x}{4x^2 - 1}$

(b) $\dfrac{x^4 - x^3 - 6x^2 + 6x + 7}{x^2 + 3x + 2}$

(c) (Revised 4/10/98) Regarding the function of part (a) as an (ordinary) generating function, what is its 17th coefficient?

(d) (Revised 4/10/98) Regarding the function of part (b) as an (ordinary) generating function, what are its first four coefficients?

Problem 5. [**20 points**]

Use generating functions to solve the recurrence

$$a_n = \begin{cases} 2^n & \text{if } n < 3 \\ 2a_{n-2} + a_{n-3} & \text{if } n \geq 3 \end{cases}$$

Use induction to check your answer.

Problem 6. [**20 points**]

Let S be the multiset $\{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \infty \cdot a_4, \infty \cdot a_5\}$. For each of the following problem parts, determine the generating function for the sequence $\langle g_0, g_1, \ldots \rangle$, where g_r is the number of r-combinations of S with the specified added restriction.

(a) Each a_k occurs an even number of times.

(b) Each a_k occurs a multiple of 5 times.

(c) The element a_1 does not occur, and a_2, a_3 occurs at most once.

(d) The element a_1 occurs 2, 5, or 8 times, and the element a_2 occurs 4, 7, or 10 times.

(e) Each a_k occurs at least 15 times.