Problem Set 7 Solutions

Problem 1.

Calculate the number of ways to place 11 *indistinguishable* balls in four *distinguishable* boxes so that each box contains at least one, but no more than four, balls.

First place one ball in each box. Then we must distribute 7 balls in four boxes in such a way that no more than three balls are in the same box.

We use inclusion/exclusion

Number of ways to place 7 balls in 4 boxes: $\binom{7+4-1}{7} = \binom{10}{7}$.

Number of ways to place 7 balls in 4 boxes such that there are four balls in a box = number of ways to choose box with four balls * number of ways to place remaining three balls in remaining three boxes $= 4 \cdot \binom{5}{3}$.

Number of ways to place 7 balls in 4 boxes such that there are five balls in a box = $4 \cdot \binom{4}{2}$.

Number of ways to place 7 balls in 4 boxes such that there are six balls in a box = $4 \cdot \binom{3}{1}$.

Number of ways to place 7 balls in 4 boxes such there are seven balls in a box = 4.

So the total number of ways is

$\binom{10}{7} - 4\left(\binom{5}{3} + \binom{4}{2} + \binom{3}{1} + 1\right) = 40$

Problem 2.

Prove that

$$\sum_{a=0}^{n} \sum_{b=0}^{n-a} \frac{n!}{a!b!(n-a-b)!} = 3^n .$$

[Hint: find a formula for $(x+y+z)^n$ and then set $x = y = z = 1$.]

We can write $(x+y+z)^n$ as follows:

$$\begin{aligned}(x+y+z)^n &= (x+y+z)(x+y+z)\cdots(x+y+z)\\ &= \sum_{\substack{a,b,c\\ a\geq 0, b\geq 0, c\geq 0\\ a+b+c=n}} C_{a,b,c}x^a y^b z^c\end{aligned} \quad . \tag{1}$$

where $C_{a,b,c}$ is the number of ways to partition a set of $a+b+c$ elements into three subsets of size, respectively, a, b, and c.

We compute $C_{a,b,c}$ using the product rule. A partition can be obtained by first choosing a elements out of $a+b+c$ ones, and then choosing b elements out of the remaining $b+c$ ones. The number of ways to make such choices is

$$C_{a,b,c} = \binom{a+b+c}{a}\cdot\binom{b+c}{b} = \frac{(a+b+c)!}{a!(b+c)!}\cdot\frac{(b+c)!}{b!c!} = \frac{(a+b+c)!}{a!b!c!} \ . \tag{2}$$

Substituting Equation (2) into Equation (1) we have

$$\begin{aligned}(x+y+z)^n &= \sum_{\substack{a,b,c\\ a\geq 0, b\geq 0, c\geq 0\\ a+b+c=n}} \frac{n!}{a!b!c!}x^a y^b z^c\\ &= \sum_{a=0}^{n}\sum_{b=0}^{n-a}\frac{n!}{a!b!(n-a-b)!}x^a y^b z^{n-a-b} \ .\end{aligned}$$

If we set $x=y=z=1$ we derive

$$3^n = (1+1+1)^n = \sum_{a=0}^{n}\sum_{b=0}^{n-a}\frac{n!}{a!b!(n-a-b)!}1^a 1^b 1^{n-a-b} \ .$$

Problem 3.

A positive integer is called *square-free* if it is not divisible by the square of a positive integer greater than 1. For example $35 = 5\cdot 7$ is square-free but $18 = 2\cdot 3^2$ is not. 1 is square-free. Find the number of square-free positive integers less than 150 (150 included.)

We use inclusion-exclusion as done in lecture for the case of prime numbers. We first compute the number of positive integers less than 150 that are not square free.

Let A_2 (respectively, A_3, A_5, A_7, A_{11}) be the set of multiples of 2^2 (respectively, 3^2, 5^2, 7^2, 11^2) less than 150. The cardinality of the union of these sets is given by

$$\begin{aligned}
|A_2 \cup A_3 \cup A_5 \cup A_7| &= |A_2| + |A_3| + |A_5| + |A_7| + |A_{11}| - |A_2 \cap A_3| - |A_2 \cap A_5| \\
&= \lfloor 150/2^2 \rfloor + \lfloor 150/3^2 \rfloor + \lfloor 150/5^2 \rfloor + \lfloor 150/7^2 \rfloor + \lfloor 150/11^2 \rfloor \\
&\quad - \lfloor 150/(2^2 3^2) \rfloor - \lfloor 150/(2^2 5^2 \rfloor \\
&= 37 + 16 + 6 + 3 + 1 - 4 - 1 \\
&= 58
\end{aligned}$$

Therefore the number of square-free positive integers less than 150 is

$$150 - 58 = 92 \ .$$

Problem 4.

(a) $4 \cdot 4 \cdot 4 \cdot 4 = 256$.

(b) 15. We can put either all 4 balls in one urn; or 3 in one, 1 in another; or 2 in one, 2 in another; or 2 in one, 1 in one and 1 in another; or 1 ball in each urn. The balls are distinguishable, so there are $\binom{4}{3}$ ways of putting 3 in one urn. There are $\binom{4}{2}$ ways of putting 2 in one and 1 in each of the other two. There are $\binom{4}{2}/2$ ways of putting 2 in one and 2 in another. Since the urns are indistinguishable, the case where all the balls are placed in the same urn and the case where each ball is places in a different urn are unique. So the total number of ways is

$$1 + \binom{4}{3} + \frac{1}{2}\binom{4}{2} + \binom{4}{2} + 1 = 15.$$

(c) 35. This is a stars and bars problem with the balls being stars and the urns bars. The answer is $\binom{4+4-1}{4} = \binom{7}{4} = 35$.

(d) 5. This is the number of ways to partition a multiset $\{1, 1, 1, 1\}$ of 4 indistinguishable objects into 4 (possibly empty) subsets. We can either put $\{1, 1, 1, 1, \}, \emptyset, \emptyset, \emptyset$ or $\{1, 1, 1\}, \{1\}, \emptyset, \emptyset$, or $\{1, 1\}, \{1, 1\}, \emptyset, \emptyset$, or $\{1, 1\}, \{1\}, \{1\}, \emptyset$, or $\{1\}, \{1\}, \{1\}, \{1\}$. The ordering in which we list the subsets doesn't matter because the urns are indistinguishable.

Problem 5.

How many ways can a photographer at a wedding arrange 5 people in a row from a group of 10 people that includes the bride and groom if

(a) The bride must be in the picture?

There are 5 ways to position the bride, $P(9,4)$ ways to choose and arrange the others. This gives a total of $\frac{5\cdot 9!}{5!}$ ways.

(b) The bride and the groom must be in the picture?

There are 5 ways to place the bride and 4 to place the groom. There are $P(8,3)$ ways to choose and arrange the people. Total : $5 \cdot 4 \cdot \frac{8!}{3!}$ ways.

(c) Exactly one of the bride and groom must be in the picture?

2 ways to choose between bride or groom, 5 ways to place whichever one is chosen, and $P(8,4)$ to choose and arrange the remaining people. Total : $2 \cdot 5\frac{8!}{4!}$ ways.

Problem 6. Six women and nine men are on the faculty of a school's EECS department. How many ways are there to select a committee of 5 members if at least 1 woman must be on the committee?

We need to count all possible combinations of people such that there is at least one woman in every combination, but we must remember not to count any combinations multiple times.

We can have committees with

1 woman, 4 men: $\binom{6}{1}\binom{9}{4}$
2 women, 3 men: $\binom{6}{2}\binom{9}{3}$
3 women, 2 men: $\binom{6}{3}\binom{9}{2}$
4 women, 1 man: $\binom{6}{4}\binom{9}{1}$
5 women, 0 men: $\binom{6}{5}\binom{9}{0}$

So there are $\binom{6}{1}\binom{9}{4} + \binom{6}{2}\binom{9}{3} + \binom{6}{3}\binom{9}{2} + \binom{6}{4}\binom{9}{1} + \binom{6}{5}\binom{9}{0} = 2877$ different possibilities for committees.

Another way to solve this problem is to say that there are $\binom{15}{5}$ different committees, and $\binom{9}{5}$ committees of just men. So there are $\binom{15}{5} - \binom{9}{5}$ different possibilities for committees.

Note that $6 \cdot \binom{14}{4} = 6006$ is not a correct answer. The product rule only applies when you are choosing two separate sets of items which are independent of each other. In this case, a woman is chosen and then the remainder of the committee is chosen, but since there may be women in the last set of 4 chosen, we will be double counting. For example, if A and B are women, and C, D, and E are men, then we will count $A, \{BCDE\}$ as well as $B, \{ACDE\}$.

Problem 7.

Let $a_1, \ldots, a_n$ be n positive integers ($n \geq 2$) such that $1 \leq a_i \leq \lfloor (2^n - 2)/n \rfloor$ for all $i = 1, \ldots, n$. Prove that there are two disjoint non-empty subsets $S, T \subseteq \{1, \ldots, n\}$ such that

$$\sum_{i \in S} a_i = \sum_{i \in T} a_i \ .$$

There are $2^n - 1$ non-empty subsets S of $\{1, \ldots, n\}$. Consider, for each such subset, the sum $sum(S) = \sum_{i \in S} a_i$. Since $1 \leq sum(S) \leq 2^n - 2$, it follows that there are two different sets S_1 and S_2 such that $sum(S_1) = sum(S_2)$ (we use the pigeonhole principle: the pigeons are the sets S and the holes are the numbers $sum(S)$; there are $2^n - 1$ pigeons and $2^n - 2$ holes.) We now define $S_1' = S_1 - (S_1 \cap S_2)$ and $S_2' = S_2 - (S_1 \cap S_2)$. We now have

$$sum(S_1') = sum(S_1) - sum(S_1 \cap S_2) = sum(S_2) - sum(S_1 \cap S_2) = sum(S_2')$$

and S_1' and S_2' are disjoint. Therefore, S_1' and S_2' are the desired solution.

Problem 8.

A shelf holds 13 books in a row. How many ways are there to choose five books so that no two adjacent books are chosen?

There are 5 bars (chosen books), and therefore there are 6 places where the 8 stars (non-chosen books) can fit (before the first bar, between the first and second bars,..., after the fifth bar). Each of the second through fifth of these slots must have at least one star in it, so that adjacent books are not chosen. Once we have placed these 4 stars, there are 4 stars left to be placed in 6 slots. The number of ways to do this is therefore $\binom{6+4-1}{4} = \binom{9}{4} = 126$.