Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Problem Set 6 Solutions

Problem 1. Consider a standard checkerboard with 64 squares arranged on an 8 by 8 grid. Suppose I place 33 checkers on the squares. Prove that either two checkers must be in one square, or two adjacent squares must both have checkers in them. Adjacent squares are those that share a full border(not just a corner). I.e., no square has more than four squares adjacent to it, and corner squares have only two squares adjacent to them.

Solution: Suppose I place the checkers on the squares. Suppose I label each square with a number between 1 and 64, putting a 1 in the upper left square, a 2 in the square to its right, a 3 in the next square to the right until I put an 8 in the upper right square, and continuing this pattern with a 9 below the 1... until I put a 64 in the lower right square. Now consider the 32 pairs of squares denoted by $2k-1, 2k$, for $1 \leq k \leq 32$. (So 1 pairs with 2, 3 pairs with 4 etc.) It is clear that for each pair of squares, the two squares in the pair are adjacent to each other.

Since there are 32 pairs and 33 checkers, then, by the pigeonhole principle, at least one pair must receive two checkers. Either both of these checkers go into the same square in that pair, or each goes into a different square in that pair, and, therefore, either two checkers go into one square or two adjacent squares(the pair) each get a checker. Therefore, if I place 33 checkers onto the squares, either two checkers must be in one square or two adjacent squares must both havfe checkers in them.

Problem 2. Find exact closed form solutions for the following recurrences.

(a) $f(n) = f(n-1) + 6f(n-2)$ for $n \geq 2$, $f(0) = 3$, $f(1) = 6$

Solution:

Characteristic equation:

$\alpha^2 - \alpha - 6 = 0$

So the roots are 3 and -2. The general solution has the form $A(-2)^n + B(3)^n$ for $A, B \in R$.

Compute A, B using the initial conditions:

$$f(0) = 3 = A(-2)^0 + B(3)^0$$

$$3 = A + B$$

$$B = 3 - A$$
$$f(1) = 6 = A(-2)^1 + B(3)^1$$
$$6 = -2A + 3B$$
$$6 = -2A + 9 - 3A$$
$$5A = 3$$
$$A = 3/5$$
$$B = 3 - A$$
$$B = 15/5 - 3/5$$
$$B = 12/5$$

So the general solution is: $f(n) = (3/5)(-2)^n + (12/5)(3)^n$.
Checking for small values of n(to avoid arithmetic and algebraic errors)

$$f(2) = 6 + 6 * 3 = 6 + 18 = 24$$

$$f(2) = (3/5)(-2)^2 + (12/5)(3)^2$$
$$f(2) = 12/5 + 108/5$$
$$f(2) = 120/5$$
$$f(2) = 24$$

The closed form is correct.

(b) $f(n) = f(n-1) + 8f(n-2) - 12f(n-3)$ for $n \geq 3$, $f(0) = 0$, $f(1) = 16$, $f(2) = 14$.

Solution:

Characteristic equation:

$\alpha^3 - \alpha^2 - 8\alpha + 12 = 0$

So the roots are -3 and a double root at 2. The general solution has the form $A(-3)^n + B(2)^n + Cn(2)^n$ for $A, B, C \in R$.

Compute A, B, and C using the initial conditions:

$$f(0) = 0 = A(-3)^0 + B(2)^0 + C(0)(2)^0$$

$$0 = A + B$$
$$A = -B$$
$$f(1) = 16 = A(-3)^1 + B(2)^1 + C(1)(2)^1$$
$$16 = -3A + 2B + 2C$$

$$
\begin{aligned}
16 &= 3B + 2B + 2C \\
16 &= 5B + 2C \\
-5B &= 2C - 16 \\
f(2) = 14 &= A(-3)^2 + B(2)^2 + C(2)(2)^1 \\
14 &= 9A + 4B + 8C \\
14 &= -9B + 4B + 8C \\
14 &= -5B + 8C \\
14 &= 2C - 16 + 8C \\
30 &= 10C \\
3 &= C \\
-5B &= 2C - 16 \\
-5B &= 6 - 16 \\
-5B &= -10 \\
B &= 2 \\
A &= -B \\
A &= -2
\end{aligned}
$$

So the general solution is: $f(n) = (-2)(-3)^n + 2(2)^n + 3n(2)^n$.
Checking for small values of n(to avoid arithmetic and algebraic errors)

$$
\begin{aligned}
f(3) &= 14 + 8 * 16 - 12 * 0 = 142 \\
f(3) &= (-2)(-3)^3 + 2(2)^3 + 3(3)(2)^3 \\
f(3) &= 54 + 16 + 72 \\
f(3) &= 142
\end{aligned}
$$

The closed form is correct.

Problem 3. Consider the following depressing scenario: Every year, each professor of computer science graduates two new students who hold a postdoc for one year and then go on to become full professors the following year. To compound the miseries of newly graduating students, in year $n \geq 3$ there are $3(n-3)$ physicists who change careers to become computer science professors without going to graduate school. Let $f(n)$ denote the number of computer science professors after n years. Assume $f(0) = 0$ and $f(1) = 1$. $f(2) = 1$: in year 2, there is one CS professor, namely the one CS professor from year 1(there are also two postdocs who graduated in year 1 and will be professors in year 3). Then $f(3) = 3$: in year 3, there are three CS professors, the one from year 1, and the two postdocs who graduated in year 1.

(a) Compute $f(4)$, $f(5)$, and $f(6)$.

Solution:

$f(4) = 3$ professors $+2$ postdocs becoming professors $+3$ physicists $= 8$.

$f(5) = 8$ professors $+6$ postdocs becoming professors $+6$ physicists $= 20$.

$f(6) = 20$ professors $+16$ postdocs becoming professors $+9$ physicists $= 45$.

(b) Set up a recurrence relation obeyed by $f(n)$(for $n \geq 3$).

Solution:

$f(n) = f(n-1) + 2f(n-2) + 3(n-3)$

(c) Solve your recurrence to find a closed form for $f(n)$ (for $n \geq 3$).

Solution:

Homogeneous Solution:

The homogeneous recurrence is:

$f(n) = f(n-1) + 2f(n-2)$.

The characteristic equation is:

$\alpha^2 - \alpha - 2 = 0 \rightarrow (\alpha+1)(\alpha-2) = 0$. So the roots are -1 and 2. The general solution is then $A(-1)^n + B(2)^n$ for $A, B \in R$.

Particular Solution: Since the "drive" formula is $3(n-3)$ we try a polynomial of the same degree (higher if 1 is a root of characteristic equation). Guess $f(n) = an + b$, and solve for a, b satisfying the recurrence:

$$an + b =$$

$$a(n-1) + b + 2(a(n-2) + b) + 3(n-3) =$$

$$an - a + b + 2an - 4a + 2b + 3n - 9 =$$

$$n(3a+3) + (3b - 5a - 9)$$

which implies that $a = -3/2$, $b = (-5a - 9)/(-2) = (15/2 - 9)/(-2) = (3/2)/(2) = 3/4$.

General Solution for Inhomogeneous Recurrence:

$f(n) = A(-1)^n + B(2)^n - (3/2)n + 3/4$

Compute A, B using $f(3)$ and $f(4)$:

$$3 = f(3) = -A + 8B - 9/2 + 3/4$$

$$-A + 8B - 9/2 + 3/4 = 3$$

$$8B = 3 + 9/2 - 3/4 + A$$

$$8B = 12/4 + 18/4 - 3/4 + A$$

$$8B = 27/4 + A$$
$$8 = f(4) = A + 16B - 6 + 3/4$$
$$A + 16B - 6 + 3/4 = 8$$
$$A + 2(27/4 + A) - 6 + 3/4 = 8$$
$$3A + 54/4 - 24/4 + 3/4 = 32/4$$
$$3A = 32/4 - 3/4 + 24/4 - 54/4$$
$$3A = -1/4$$
$$A = -1/12$$
$$8B = 27/4 - 1/12$$
$$8B = 81/12 - 1/12$$
$$8B = 80/12$$
$$B = 10/12$$

$f(n) = (-1/12)(-1)^n + (10/12)(2)^n - (3/2)n + 3/4$
Checking these for small values of n (to avoid arithmetic and algebraic errors):

$$f(3) = (-1/12)(-1)^3 + (10/12)(2)^3 - (3/2)3 + 3/4$$
$$f(3) = 1/12 + 80/12 - 9/2 + 3/4$$
$$f(3) = 1/12 + 80/12 - 54/12 + 9/12$$
$$f(3) = 36/12 = 3$$

$$f(4) = (-1/12)(-1)^4 + (10/12)(2)^4 - (3/2)4 + 3/4$$
$$f(4) = -1/12 + 160/12 - 12/2 + 3/4$$
$$f(4) = -1/12 + 160/12 - 72/12 + 9/12$$
$$f(4) = 96/12 = 8$$

$$f(5) = (-1/12)(-1)^5 + (10/12)(2)^5 - (3/2)5 + 3/4$$
$$f(5) = 1/12 + 320/12 - 15/2 + 3/4$$
$$f(5) = 1/12 + 320/12 - 90/12 + 9/12$$
$$f(5) = 240/12 = 20$$

The closed form is correct.

Problem 4. Give orders of growth for the following divide-and-conquer recurrences:

(a) $f(n) = 2f(n/2) + 3$
Solution: Using the Master Theorem, $f(n) = af(n/b) + cn^d$, $a = 2$, $b = 2$, $c = 3$, $d = 0$, $b^d = 1$, so $a = 2 > b^d = 1$, so $f(n) = \Theta(n^{log_b a})$, so $f(n) = \Theta(n^1) \to f(n) = \Theta(n)$

(b) $f(n) = 2f(n^{\frac{1}{3}}) + n$
Solution: First perform a change of variables. Set $n = 2^m$ and therefore $m = log_2 n$. Then

$$f(2^m) = 2f(2^{m^{\frac{1}{3}}}) + 2^m$$
$$f(2^m) = 2f(2^{\frac{m}{3}}) + 2^m$$

Set $S(m) = f(2^m)$. Then

$$S(m) = 2S(m/3) + 2^m$$

This would seem to fall in case 3 of the master theorem. Thus we need to find $c < 1$ such that $2 * 2^{(\frac{m}{3})} < c * 2^m$ for large m. This means that:

$$2^{\frac{m}{3}+1} < c * 2^m$$
$$\frac{m}{3} + 1 < c * m$$

Pick $c = \frac{1}{2}$ and this is certainly true for $m > 6$. Applying the Master theorem, $S(m) = \Theta(2^m)$, so $f(n) = \Theta(n)$.

Problem 5. For each of the following functions, $f : \mathbb{R} \to \mathbb{R}$, determine whether it is an injection, surjection, or bijection and give the inverse if it exists. (The inverse of a function is defined in Rosen page 64).

(a) $f(x) = sin(x)$
Solution: $f(x) = sin(x)$ is not an injection because it maps 0 and π to 0. It is not a surjection because it maps $\mathbb{R}$ onto $[-1, 1]$. Therefore it is not a bijection and does not have an inverse.

(b) $f(x) = \frac{x+2}{x^2+4}$
Solution: $f(x) = \frac{x+2}{x^2+4}$ is not an injection because it maps 0 and 2 to $\frac{1}{2}$. It is not a surjection because 1 has no pre-image. $x^2 + 4 = x + 2$ has no real solutions. Therefore it is not a bijection and does not have an inverse.

(c) $f(x) = \begin{cases} \ln x & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ e^x & \text{if } x < 0 \end{cases}$

Solution: This function is not an injection because $ln\frac{1}{2}$ and $e^{\frac{1}{2}}$ both map to $\frac{1}{2}$. It is a surjection because for any $x \in \mathbb{R}$, a pre-image of x is e^x for $x \neq 0$, and 0 for $x = 0$.

(d) $f(x) = x^3 + 2$

Solution: $f(x) = x^3 + 2$ is an injection because $x \neq y \to x^3 \neq y^3 \to x^3 + 2 \neq y^3 + 2$. The function is a surjection because for every $x \in \mathbb{R}$ $\exists y = (x-2)^{\frac{1}{3}}$. Therefore f is a bijection. It's inverse is $(x-2)^{\frac{1}{3}}$.

Problem 6. Consider the set of binary strings of infinite length (strings of zeroes and ones). Prove that the set of such strings with at most 501 ones in them is countable. (Hint: how might you list these numbers?)

Solution: Let A be the set of infinite binary strings with at most 501 ones in them. For any arbitrary $x > 0$, only a finite number of elements of A have all of their ones occur before the xth digit. (i.e., only 2 have all ones before the 2nd digit, specifically $1000000\ldots$ and $000000\ldots$) Let A_x be the set of those elements in A such that all of their ones occur before the xth digit. Each A_x is a finite set, and a subset of A. For any element $a \in A$, look at the position of the last one in a. If that one is in position i then $a \in A_{i+1}$. Therefore, every element of A appears in at least one A_x. To list all of the elements in A, first list all of the elements in A_1, then all of the elements in A_2, then all of the elements in A_3 etc, removing duplicates. Since each A_x is finite, the list is a valid list. Thus a list has been generated for all elements in A and therefore A is countable.

Problem 7. Consider the set of binary strings of infinite length (strings of zeroes and ones). Define A_n to be the set of binary strings such that, at any point in the string, at most $1/n$ of the preceding numbers can be 1. So A_1 is the set of all binary strings of infinite length, since at any point, at most all of the preceding numbers can be 1. A_2 is the set of strings such that at any point, at most half of the preceding terms can be 1. For instance, the first element of every string in A_2 must be 0, and of the first three elements, at most 1 can be 1. In general, the first $n-1$ elements of A_n must be 0. As a further example, the string 010101000... $\in A_2$, but the string 0101011000... $\notin A_2$.

Prove that A_{501} is uncountable.

Solution: The proof is by diagonalization. Assume for the purposes of contradiction that the set is countable. Then there must be a way to list the elements in order. Construct such a list, using a mapping function such that $f(i)$ represents the ith element in the list. Now construct a number a as follows. For each $f(i)$, set the $501(i)$th element of a to be the opposite of the $501(i)$th element of $f(i)$. For all elements of a not set in this manner, set them to 0. Since at any point, there are at least 500 zeroes for every one, a is in A_{501}. Therefore it must appear in this list at $a = f(j)$. However, $f(j)$ and a differ on the $501(j)$th element of the string, so $a \neq f(j)$, and a contradiction is reached. Therefore a cannot be in the list and the elements of A_{501} cannot be listed, therefore A_{501} is uncountable.