Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Handout 30
March 17, 1998

Problem Set 6

Due: Tuesday, March 31.

Reading assignment: Rosen sections 1.6-1.7, 5.1-5.3

Problem 1. Consider a standard checkerboard with 64 squares arranged on an 8 by 8 grid. Suppose I place 33 checkers on the squares. Prove that either two checkers must be in one square, or two adjacent squares must both have checkers in them. Adjacent squares are those that share a full border(not just a corner). I.e., no square has more than four squares adjacent to it, and corner squares have only two squares adjacent to them.

Problem 2. Find exact closed form solutions for the following recurrences.

(a) $f(n) = f(n-1) + 6f(n-2)$ for $n \geq 2$, $f(0) = 3$, $f(1) = 6$.

(b) $f(n) = f(n-1) + 8f(n-2) - 12f(n-3)$ for $n \geq 3$, $f(0) = 0$, $f(1) = 16$, $f(2) = 14$.

Problem 3. Consider the following depressing scenario: Every year, each professor of computer science graduates two new students who hold a postdoc for one year and then go on to become full professors the following year. To compound the miseries of newly graduating students, in year $n \geq 3$, there are $3(n-3)$ physicists who change careers to become computer science professors without going to graduate school. Let $f(n)$ denote the number of computer science professors after n years. Assume $f(0) = 0$ and $f(1) = 1$. It follows that $f(2) = 1$: in year 2, there is one CS professor, namely the one CS professor from year 1 (there are also two postdocs who graduated in year 1 and will be professors in year 3). Then $f(3) = 3$: in year 3, there are three CS professors, the one from year 1, and the two postdocs who graduated in year 1.

(a) Compute $f(4)$, $f(5)$, and $f(6)$.

(b) Set up a recurrence relation obeyed by $f(n)$(for $n \geq 3$).

(c) Solve your recurrence to find a closed form for $f(n)$(for $n \geq 3$).

Problem 4. Give orders of growth for the following divide-and-conquer recurrences:

(a) $f(n) = 2f(n/2) + 3$

(b) $f(n) = 2f(n^{\frac{1}{3}}) + n$

Problem 5. For each of the following functions, $f : \mathbb{R} \to \mathbb{R}$, determine whether it is an injection, surjection, or bijection, and give the inverse if it exists. (The inverse of a function is defined in Rosen page 64).

(a) $f(x) = sin(x)$

(b) $f(x) = \frac{x+2}{x^2+4}$

(c) $f(x) = \begin{cases} lnx & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ e^x & \text{if } x < 0 \end{cases}$

(d) $f(x) = x^3 + 2$

Problem 6. Consider the set of binary strings of infinite length (strings of zeroes and ones). Prove that the set of such strings with at most 501 ones in them is countable. (Hint: how might you list these numbers?)

Problem 7. Consider the set of binary strings of infinite length (strings of zeroes and ones). Define A_n to be the set of binary strings such that, at any point in the string, at most $1/n$ of the preceding numbers can be 1. So A_1 is the set of all binary strings of infinite length, since at any point, at most all of the preceding numbers can be 1. A_2 is the set of strings such that at any point, at most half of the preceding terms can be 1. For instance, the first element of every string in A_2 must be 0, and of the first three elements, at most 1 can be 1. In general, the first $n-1$ elements of A_n must be 0. As a further example, the string 010101000... $\in A_2$, but the string 0101011000... $\notin A_2$.

Prove that A_{501} is uncountable.