Problem Set 5 Solutions

Problem 1. A college loan officer offers prospective M.Eng students the following deal. He will give them \$23,100 at the start of each year for five years starting today to pay for their college tuition. He offers the following plan for repayment:
Wait five years, then repay \$$x$.00 at the start of each year for the next ten years.

Suppose the annual interest rate is 8%. What value of x will make the present value of the repayment equal to the present value of the loan?

The formula for finding the present value of an annuity (given in lecture) is

$$V = m \sum_{j=0}^{n-1} \frac{1}{(1+p)^j}$$

where V is the present value, m is the dollar amount of each payment, n is the number of years of the annuity, and $p = 0.08$ is the current interest rate.

So the present value of what the college loan officer offers is

$$\begin{aligned} V &= 23100 \cdot \sum_{j=0}^{4} (\frac{1}{1+p})^j \\ &= 23100 \cdot \frac{(1 - \frac{1}{(1+p)^5})}{(1 - \frac{1}{1+p})} \\ &= 23100 \cdot 4.312 \\ &= \$99,610.13. \end{aligned}$$

The value, V', of the repayment is

$$V' = x \cdot \sum_{j=5}^{14} \frac{1}{(1+p)^j}.$$

Since we want the present values to be equal, let $V' = V$, and solve for x:

$$\begin{aligned} x &= \frac{\$99,610.13}{(\frac{1}{1+p})^5 \cdot (\sum_{j=0}^{9} (\frac{1}{1+p})^j)} \\ &= \frac{\$99,610.13}{(\frac{1}{1+p})^5 \cdot \frac{(1-\frac{1}{(1+p)^{10}})}{(1-\frac{1}{1+p})}} \\ &= \frac{\$99,610.13}{0.6805 \cdot 7.246} \\ &= \$20,196.25. \end{aligned}$$

So, rounding off to the nearest dollar, $x = \$20,196$.

Problem 2. **(a)** Find a closed form solution for

$$\sum_{k=1}^{n} 1/k(k+1)$$

Hint: Use the fact that $1/k(k+1) = 1/k - 1/(k+1)$.

We will use the fact that

$$\sum_{j=0}^{n-1} (a_j - a_{j+1}) = a_0 - a_n,$$

where $a_0, a_1, ..., a_n$ is a sequence of real numbers. (This type of sum is called *telescoping*, for obvious reasons). This holds almost by inspection: we can say that the $-a_{j+1}$ in one term of the summation will be the a_j in the subsequent term, and the only a_j's that don't cancel out in this manner are a_0, since there is no term before $(a_0 - a_1)$ in the summation, and a_n, since there is no term after $(a_{n-1} - a_n)$ in the summation. Formally, this equation can be verified by an easy induction which we omit.

Now (to get back to the problem at hand), using the given hint,

$$\sum_{k=1}^{n} \frac{1}{(k(k+1))} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{(k+1)}$$

Letting $a_i = \frac{1}{i+1}$, for $0 \le i \le n-1$, the righthand sum above is

$$\sum_{i=0}^{n-1} (a_i - a_{i+1}) = a_0 - a_n.$$

Now

$$\begin{aligned} a_0 - a_n &= \frac{1}{0+1} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1} \\ &= \frac{n}{n+1}. \end{aligned}$$

(b) Use induction to verify your closed form.

Proof by induction. Let $P(n)$ be the statement that

$$\forall n > 1, \sum_{k=1}^{n} \frac{1}{(k(k+1))} = \frac{n}{n+1}$$

Base Case. P(1) is true since

$$\sum_{k=1}^{1} \frac{1}{(k(k+1))} = \frac{1}{2} = \frac{1}{1+1}$$

.

Inductive Step. Assume $P(n)$ is true. Then

$$\begin{aligned} \sum_{k=1}^{n+1} \frac{1}{(k(k+1))} &= \frac{1}{(n+1)((n+1)+1)} + \sum_{k=1}^{n} \frac{1}{(k(k+1))} \\ &= \frac{1}{(n+1)(n+2)} + \frac{n}{n+1} \\ &= \frac{n^2+2n+1}{(n+1)(n+2)} \\ &= \frac{(n+1)^2}{(n+1)(n+2)} \\ &= \frac{n+1}{((n+1)+1)} \end{aligned}$$

Therefore by induction,

$$\forall n > 1, \sum_{k=1}^{n} \frac{1}{(k(k+1))} = \frac{n}{n+1}$$

Problem 3. Find a closed form solution for $\prod_{i=1}^{n} 2 \cdot 4^i$.

We know that $\prod_{i=1}^{n} 2 \cdot 4^i = (2 \cdot 4^1) \cdot (2 \cdot 4^2) \cdot \ldots \cdot (2 \cdot 4^n)$. Taking $\log_2$ of both sides of the equation gives us

$$\begin{aligned}
\log_2(\prod_{i=1}^{n} 2 \cdot 4^i) &= \log_2(2 \cdot 4^1 \cdot 2 \cdot 4^2 \cdot \ldots \cdot 2 \cdot 4^n) \\
&= \log_2(2 \cdot 4^1) + \log_2(2 \cdot 4^2) + \ldots + \log_2(2 \cdot 4^n) \\
&= \sum_{j=1}^{n} \log_2(2 \cdot 4^j) \\
&= \sum_{j=1}^{n} (\log_2(2) + \log_2(4^j)) \\
&= \sum_{j=1}^{n} (1 + 2j) \\
&= n + n(n+1) \\
&= n^2 + 2n.
\end{aligned}$$

Now, exponentiating both sides of the equation gives $\prod_{i=1}^{n} 2 \cdot 4^i = 2^{n^2+2n}$

Problem 4. Let $0 < x < 1$ and $0 < y < 1$. Evaluate the following infinite sums.

(a)

$$\begin{aligned}
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x^i y^{i+2j} &= \sum_{i=1}^{\infty} [(xy)^i (\sum_{j=1}^{\infty} (y^2)^j)] \\
&= \sum_{i=1}^{\infty} [(xy)^i (y^2/(1-y^2))] \\
&= [\sum_{i=1}^{\infty} (xy)^i] y^2/(1-y^2) \\
&= [xy/(1-xy)] y^2/(1-y^2) \\
&= (xy^3)/(1-xy)(1-y^2).
\end{aligned}$$

(b)

$$\begin{aligned}
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} ijx^i y^j &= \sum_{i=1}^{\infty} [ix^i (\sum_{j=1}^{\infty} jy^j)] \\
&= [\sum_{i=1}^{\infty} ix^i] (\sum_{j=1}^{\infty} jy^j) \\
&= [x/(1-x)^2] y/(1-y)^2 \\
&= (xy)/((1-x)(1-y))^2
\end{aligned}$$

Problem 5. Justify your answers to the following:

(a) Is 2^n $O(4^n)$?
Yes, $2^n = O(4^n)$. To show this, we need to find an n_0 and a c that satisfy the definition of O, specifically, $\forall n > n_0, 2^n < c \cdot 4^n$. Choose $n_0 = 0$ and $c = 1$. Then we have $\forall n > 0, 2^n < 4^n$, which is certainly true.

(b) Is 2^{2n} $O(2^n)$?
2^{2n} is not $O(2^n)$. To prove that $2^{2n} = O(2^n)$, we have to find an n_0 and c for which $\forall n > n_0, 2^{2n} < c \cdot 2^n$. Let's divide both sides by 2^n, which we can do since 2^n is always positive, and never equal to 0. Then we have $\forall n > n_0, 2^n < c$. Since 2^n is unbounded as $n \to \infty$, there's no n_0 for which 2^n will be less than a constant.

(c) Is 2^{n+1} $O(2^n)$?
Yes, $2^{n+1} = O(2^n)$, with (for example) $c = 2$, and $n_0 = 0$.

(d) Is $\log_{10} x = \Theta(\log_2 x)$?
Yes. The change of base formula tells us that $\log_a n = \frac{\log_c n}{\log_c a}$. Therefore, we know that $\log_{10} n = \frac{\log_2 n}{\log_2 10}$. For all $n > 0$, $\log_{10} n$ is within a constant factor, namely $\frac{1}{\log_2 10}$, of $\log_2 n$.

Problem 6. Find a Θ bound for $\sum_{k=1}^{n} k^6$.

Let $\sum_{k=1}^{n} k^6 = S_n$.
There are n terms in S_n and each term is at most n^6, so $S_n \leq n \cdot n^6 = n^7 = O(n^7)$. So $S_n = O(n^7)$.
On the other hand, at least $(n-1)/2$ of the terms are as large as $[(n-1)/2]^6$, so

$$\begin{aligned} S_n &\geq ((n-1)/2) \cdot [(n-1)/2]^6 \\ &= [(n-1)/2]^7 \\ &\geq (n/3)^7 \end{aligned}$$

for $n > 3$, so $n^7 \leq 3^7 \cdot S_n$. In other words, $n^7 = O(S_n)$. Since we've shown $S_n = O(n^7)$ and $n^7 = O(S_n)$, we know that $S_n = \Theta(n^7)$.
We can also do this problem using the Integral Method:

$$\int_0^n x^6 \, dx \leq \sum_{k=1}^{n} k^6 \leq \int_0^n (x+1)^6 \, dx$$
$$\frac{n^7}{7} \leq \sum_{k=1}^{n} k^6 \leq \frac{(n+1)^7}{7} - \frac{1}{7}$$

From which we see that $\sum_{k=1}^{n} k^6 = \Theta(n^7)$

Problem 7. Prove or give a counterexample: If $f(n) = \Theta(n)$, then $\sum_{i=1}^{n} f(i) = \Theta(n^2)$.

If $f(n) = \Theta(n)$, we know that $\exists c_1, c_2, k > 0$, such that $\forall n > k, c_1 n \leq f(n) \leq c_2 n$. Let's assume that $n > 2k$. (Note that we can make assumptions of this type since we're free to choose the lower bound on n needed in the definition of the Θ bound that we're proving. That is, we're going to prove that $a_1 n^2 \leq \sum_{i=1}^{n} f(i) \leq a_2 n^2$ for all $n > n_0$, and we can make the n_0 as large as we'd like.)

Let's look at $\sum_{i=k+1}^{n} c_1 i$. Since $n > 2k$, at least $\frac{n}{2}$ terms in the sum must be bigger than $c_1 \frac{n}{2}$, which we can state mathematically as follows:

$$\begin{aligned}
(\frac{n}{2})^2 \cdot c_1 &\leq \sum_{i=k+1}^{n} c_1 i \\
&\leq \sum_{i=k+1}^{n} f(i) \\
&\leq \sum_{i=k+1}^{n} c_2 i \\
&\leq \sum_{i=k+1}^{n} c_2 n \\
&\leq c_2 n^2.
\end{aligned}$$

So,

$$\frac{c_1}{4} n^2 \leq \sum_{i=k+1}^{n} f(i) \leq c_2 n^2$$

Now we need to manipulate this to get the summation to start at 1, rather than $k+1$. (Note that the definition of Θ will only let us assume that $c_1 n \leq f(i) \leq c_2 n$ for $i > k$, so we need to find some other method of dealing with the sum from 1 to k).

Let $s = \sum_{i=1}^{k} f(i)$. Then, by adding s to each part of the inequality above, we get

$$\frac{c_1}{4} n^2 + s \leq \sum_{i=1}^{n} f(i) \leq c_2 n^2 + s$$

Now, assume that $n \geq \sqrt{|s|}$, so we can write $c_2 n^2 + s \leq (c_2 + 1) n^2$

Assume $n \geq \sqrt{8\,|s|\,/c_1}$, so $\frac{c_1}{8} n^2 \leq \frac{c_1}{4} n^2 + s$

So, letting $n_0 = max(2k, \sqrt{|s|}, \sqrt{8\,|s|\,/c_1})$, we now know that for all $n > n_0$,

$$\frac{c_1}{8}n^2 \le \frac{c_1}{4}n^2 + s \le \sum_{i=1}^{n} f(i) \le c_2 n^2 + s \le (c_2+1)n^2,$$

which satisfies the definition of $\Theta(n^2)$, with $a_1 = \frac{c_1}{8}$, $a_2 = c_2 + 1$. ∎

Problem 8. **(a)** Find a closed form solution for $\sum_{i=1}^{n} i^2 x^i$.

From lecture we know that

$$\sum_{i=1}^{n} ix^i = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}.$$

Differentiating (with respect to x) the above expression and multiplying with x we find that

$$\begin{aligned} \sum_{i=1}^{n} i^2 x^{i-1} &= \frac{1 + x - (n+1)^2 x^n + (2n^2+2n-1)x^{n+1} - n^2 x^{n+2}}{(1-x)^3} \\ \sum_{i=1}^{n} i^2 x^{i} &= \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} \end{aligned}$$

(b) Verify your closed form by induction.

Proof by induction. Let $P(n)$ be the statement that

$$\sum_{i=1}^{n} i^2 x^{i} = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}.$$

Base Case. $P(1)$ is true since

$$\sum_{i=1}^{1} i^2 x^i = x = \frac{x + x^2 - 4x^2 + 3x^3 - x^4}{(1-x)^3}.$$

Inductive Step. Assume $P(n)$ is true. Then

$$\begin{aligned} \sum_{i=1}^{n+1} i^2 x^i &= \sum_{i=1}^{n} i^2 x^i + (n+1)^2 x^{n+1} \\ &= \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} + (n+1)^2 x^{n+1} \\ &= \frac{x + x^2 - (n+2)^2 x^{n+2} + (2(n+1)^2+2(n+1)-1)x^{n+3} - (n+1)^2 x^{n+4}}{(1-x)^3} \end{aligned}$$

Thus $P(n+1)$ is true. Hence, $P(n)$ is true for all n.

Problem 9. Let $S_n = \sum_{i=1}^{n} i^{1/3}$. Use the integral method to determine

$$\lim_{n\to\infty} \frac{S_n}{n^{4/3}}$$

We use the integral method for bounding S_n.

$$\int_0^n x^{\frac{1}{3}} dx \leq S_n \leq \int_1^{n+1} x^{\frac{1}{3}} dx.$$

That is,

$$\frac{(3/4)n^{4/3}}{n^{4/3}} \leq \frac{S_n}{n^{4/3}} \leq \frac{(3/4)((n+1)^{4/3}-1)}{n^{4/3}}$$

Taking limits as $n \to \infty$ we find that

$$3/4 \leq \lim_{n\to\infty} \frac{S_n}{n^{4/3}} \leq 3/4$$

Therefore, $\lim_{n\to\infty} \frac{S_n}{n^{4/3}} = 3/4$.