Problem Set 3 Solutions

Reading assignment: Rosen sections 7.1–7.4, 7.8, 8.1–2

Problem 1. [10 pts]

The graphs C_n, Q_n, W_n, and $K_{n,m}$ are defined in Rosen 7.2.

(a) Describe the adjacency matrix for C_n.

For C_n, we label the vertices so that the cycle goes $v_1, v_2, \ldots, v_n, v_1$. Then the matrix has 1's on the diagonals just above and below the main diagonal and in positions $(1, n)$ and $(n, 1)$, and 0's everywhere else.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

(b) Describe the adjacency matrix for $K_{n,m}$.

Vertices of $K_{n,m}$ are partitioned into a group of n vertices and a group of m vertices. In this graph, an edge is present between two vertices if and only if the two vertices are from different groups. Therefore, the adjacency matrix of $K_{n,m}$ consists of four sub-matrices. The top left and the bottom right matrices consist of 0's with size $n * n$ and $m * m$ respectively. The top right and the bottom left matrices consist of 1's with size of $n * m$ and $m * n$ respectively.

$$\begin{pmatrix} 0 & \ldots & 0 & 1 & \ldots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & \ldots & 1 \\ 1 & \ldots & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \ldots & 1 & 0 & \ldots & 0 \end{pmatrix}$$

(c) Give a simple recursive definition of the adjacency matrix for the graph Q_{n+1}. *Hint*: Use the matrix of Q_n and suitable identity matrices.

The base case is,

$$Q_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then, every Q_{n+1} ($n \geq 1$) is defined inductively. The size of Q_{n+1} is $2^{n+1} * 2^{n+1}$ and the left top and right bottom quarter of the matrix are Q_n, which correspond to the two subgraphs Q_n in Q_{n+1}. The right top and left bottom quarter of the matrix are identity matrices of size $2^n * 2^n(I_{2^n})$, which correspond to the edges connecting vertices k and $k + 2^n$ for $0 \leq k < 2^n$, which differ by the most significant bit only.

$$Q_{n+1} = \begin{pmatrix} Q_n & I_{2^n} \\ I_{2^n} & Q_n \end{pmatrix}$$

Problem 2. [10 pts]

Find the minimum number of colors to color (with no two adjacent vertices given the same color, *cf.* Rosen, 7.8) each of C_9, C_{20}, W_{13} and $K_{311,222}$. Explain briefly.

C_9 and C_{20} can be colored with 3 and 2 colors respectively. To see this, let's number the vertices of C_n from 1 to n by going around the cycle. If n is even, we can assign every even vertex with one color and every odd vertex with the other color. Since n is even and 1 is odd, vertex n and vertex 1 will have different colors. Therefore, we only need 2 colors.

If n is odd, we need one more color, since both 1 and n are odd and they cannot have the same color. Therefore, we will assign all the even vertices with one color, all the odd vertices except n with a second color and n with a third color. Therefore, we need 3 colors. Since 9 is odd and 20 is even, C_9 and C_{20} need 3 colors and 2 colors respectively.

Wheels can be colored with very few colors as well. A wheel is indeed a cycle with one extra vertex adjacent to all the vertices in the cycle. Therefore we only need one more color for the extra vertex. Since we need 2 colors for an even cycle and 3 colors for an odd cycle, an even wheel needs 3 colors and an odd wheel needs 4 colors. Therefore, we can color W_{13} with 4 colors.

The number of colors for $K_{311,222}$ seems to depend on 311 and 222. However, only two colors are needed. Color the set of the 311 vertices with one color and the set of the 222 vertices with a second color. Since no edge connects two vertices in the same group in a bipartite graph, no two adjacent vertices have the same color. In fact, we can extend this result to any m and n in $K_{m,n}$. Therefore, any complete bipartite graph is 2-colorable.

Problem 3. [20 pts]

(a) Prove or disprove that the pair of graphs given in Rosen 7.3, Exer. 38 are isomorphic.

The two graphs are isomorphic. Informally, we can see by breaking the graph into a K_4 and a extra vertex with two edges connected to K_4. Therefore, a function f that maps $u_1 \to v_1$, $u_2 \to v_3$, $u_3 \to v_2$, $u_4 \to v_5$ and $u_5 \to v_4$, is an isomporphism between the two graphs. We can verify this since there is an edge between u_i and u_j iff there is an edge between $f(u_i)$ and $f(u_j)$ for all $1 \leq i, j \leq 5$.

(b) Prove or disprove that the pair of graphs given in Rosen 7.3, Exer. 42 are isomorphic.

The two graphs are not isomorphic. Both graphs have two vertices of degree 3: u_2 and u_3 in the first graph and v_2 and v_4 in the second graph. Since u_2 and u_3 are adjacent while v_2 and v_4 are not, the two graphs are not isomorphic.

Problem 4. [30 pts]

Some properties of a simple graph, G, are described below. For each property, carefully prove or give a counterexample to the claim that the property is *invariant under isomorphism* (*cf.* Rosen, p.455).

(a) G has an even number of vertices.

It is invariant under isomorphism. There must be an one-to-one and onto mapping between the vertices of two isomorphic graphs. Therefore, the number of vertices in the two graphs must be the same. If one graph has even number of vertices, then the other must have even number of vertices.

(b) None of the vertices of G is an even integer.

It is not invariant under isomorphism. We do not really care what the vertices are. Vertices can be any kind of mathematical objects. All we are interested is that whether there exists a one-to-one and onto function f mapping from vertices of one graph to vertices of another with the property that a and b are adjacent in the first graph if and only if $f(a)$ and $f(b)$ are adjacent in the second graph, for all a and b in the first graph.

So, for example, let G_1 be a graph with a single vertex, 1, and G_2 be a graph with a single vertex, 2. Obviously the two graphs are isomorphic, but G_2 does not have vertices which are even integers.

(c) G has a vertex of degree 3.

It is invariant under isomorphism.

Let G_1, G_2 be simple graphs and $f : V_1 \to V_2$ be an isomorphism between them. Suppose $v \in V_1$ has degree 3; we want to show that there is a vertex of degree 3 in V_2. In fact, we'll show that $f(v)$ has degree 3.

Since v has degree 3, there are exactly 3 vertices adjacent to v; say these are v_1, v_2, v_3. Since f is a bijection, $f(v_1), f(v_2)$ and $f(v_3)$ are all distinct. Since there is an edge between v and v_i in G_1, the definition of isomrphism implies that there is an edge between $f(v)$ and $f(v_i)$ for $i = 1, 2, 3$, so the degree of $f(v)$ is at least 3.

We now prove by contradiction that the degree of $f(v)$ is at most 3. Suppose $f(v)$ had degree > 3. This means there is a vertex $w \in V_2$ which is not equal to $f(v_1), f(v_2)$, or $f(v_3)$, but is also adjacent to $f(v)$. Since, f is a bijection, there is a vertex $v_4 \in V_1$ such that $f(v_4) = w$ and $v_4 \neq v_i$ for $i = 1, 2, 3$. Since $w = f(v_4)$ is adjacent to $f(v)$, the definition of isomorphism implies that v_4 is adjacent to v, contradicting the fact the v_1, v_2, v_3 were exactly the vertices adjacent to v.

(d) G has exactly one vertex of degree 3.

It is invariant under isomorphism.

Prove by contradiction: Let a graph G_1 has exactly one vertex of degree 3 while another graph G_2 does not have exactly one vertex of degree 3. Suppose the two graphs are isomorphic. If G_2 does not have a vertex of degree 3, then from part (c), there is a contradiction. If G_2 has more than one vertices of degree 3, then there must be at least one vertex in G_2 of degree 3 which is mapped to a vertex of degree $\neq 3$ in G_1. Since two vertices of different degrees are mapped from G_1 to G_2, using the same argument from part (c), it reaches a contradiction. Therefore, if G_1 has exactly one vertex of degree 3 , then G_2 must also have exactly one vertex of degree 3.

(e) One of the vertices of G is also an edge of G.

It is not invariant under isomorphism.

Let G_1 be a graph with three vertices, 1, 2 and $\{1, 2\}$. There is only one edge from vertex 1 to vertex 2. Therefore, $\{1, 2\}$ is a vertex of G_1 and also an edge of G_1.

On the other hand, let G_2 be a graph with three vertices, 7, 12.1 and an ordered pair $(1, -3)$(note: these are somewhat tasteless, but legitimate things to use as vertices). There is only one edge from vertex 7 to vertex 12.1. Obviously two graphs are isomorphic; however, G_2 does not have a vertex which is an edge at the same time. Therefore, it is not invariant under isomorphism.

Problem 5. [30 pts]

Definition: A simple graph, G, has *inherited degree* $d \geq 0$ if and only if
(i) every vertex of G has degree $\leq d$, or
(ii) there is a vertex $v \in G$ of degree $\leq d$, and the subgraph of G obtained by removing v has inherited degree $\leq d$.

(a) Prove that W_n has inherited degree 3.

We use the induction on n. Let $P(n)$::= W_n has inherited degree 3.

Base Case: W_3 has four vertices of degree 3 and it has inherited degree 3 by definition (i).

Inductive Step: Assume $P(n)$ is true, we need to prove $P(n+1)$. W_{n+1} has $n+1$ vertices of degree 3 and 1 vertex of degree $n+1$. The graph G obtained by removing a vertex of degree 3 has $n-2$ vertices of degree 3, 2 vertices of degree 2 and 1 vertex of degree n. We can also obtain G by removing a edge between two vertices of degree 3 in W_n. Since W_n has inherited degree 3 by inductive hypothesis and removing an edge does not increase the degree of vertices, G has inherited degree 3. Then, by definition (ii), W_{n+1} has inherited degree 3.

Therefore, we prove that W_n has inherited degree 3 for $\forall n \geq 3$.

(b) Prove that a tree has inherited degree 1.

We use the definition of tree in Rosen, 8.1: a tree is a connected undirected graph with no simple circuits. Also, we only consider trees with a finite number of vertices.

We need to prove two facts here. First we have to prove that every tree with at least two vertices has a leaf (a leaf is a vertex of degree one). Then we need to show that the subgraph constructed by removing one leaf is still a tree.

First, let's prove that every tree with at least two vertices has a leaf. Let T be a tree, and let P be a longest simple path in T. In a finite graph, a path of length greater than or equal to the the number of vertices must visit some vertex more than once. So a *simple* path must be shorter than the number of vertices. Therefore in any finite graph there must be longest simple path. (We're using the Well-ordering principle here, taking the least element of $\{|V| - k : k \text{ is the length of a simple path}\}$.)

Suppose $P = (v_1, v_2 \ldots, v_k)$, we claim that v_1 is a leaf. For if not, there will be an edge $\{v, v_1\}$ in T where $v \neq v_2$. The vertex v cannot occur in P, for otherwise this creates a cycle, which contradicts that the graph is a tree. Therefore, $P' = (v, v_1, v_2 \ldots, v_k)$ is also a simple path in the tree and P' is longer than P. But this is a contradiction because we have assumed that P is the longest path in the tree. As a result, a tree with 2 or more vertices has to have a leaf.

Second, let's prove that the subgraph constructed by removing one leaf is still a tree. From the definition given in the book (cf. Rosen p531), a tree is a connected undirected graph with no simple circuits. Removing a leaf from a tree does not create any circuit. Also, since the leaf has degree 1, it is not connecting any other two vertices in the tree. Therefore, after removing the leaf, no vertices get disconnected. In other words, the subgraph is still connected. Since the subgraph is a connected undirected graph with no simple circuits, it is also a tree.

Since every tree has a leaf(vertex with degree 1) and the subgraph is still a tree after removing a leaf, we can apply the same method in the previous part to show that a tree has inherited degree 1.

(c) Prove that if a graph has inherited degree d, then it is $(d+1)$-colorable.

We use induction on the number of vertices in a graph to prove this fact.

Let $P(n)$::= "$\forall d > 0$, any n-vertex graphs with inherited degree d is $(d+1)$ colorable."

Base Case: $n = 1$. There is only one 1-vertex graph. And it is obviously $(d+1)$ colorable ($\forall d > 0$).

Inductive Step: Assume that $P(n)$ is true, we need to prove $P(n+1)$ is true. Given any $(n+1)$-vertex graph G with inherited degree d for any $d > 0$, there are 2 possible cases:

1. All vertices in G has degree $\leq d$. Then it is $(d+1)$-colorable, as proved in the lecture.
2. There is a vertex $v \in G$ of degree $\leq d$, and the subgraph G' obtained by removing v has inherited degree d. Since G' has n nodes and has inherited degree d, by inductive hypothesis G' is $(d+1)$ colorable. Now consider the vertex v. Since it has degree $\leq d$, it has at most d neighbors and therefore it can be colored with the $(d+1)^{st}$ color which is different from all of its neighbors.

Therefore, by induction, every graph with inherited degree d is $(d+1)$-colorable.

(d) For every $n > 0$, describe an example of a graph which does not have inherited degree n, but which is 2-colorable.

An example would be $K_{n+1,n+1}$. Every vertex in $K_{n+1,n+1}$ has degree $n+1$. Therefore, it does not have inherited degree n. However, it is 2-colorable since every complete bipartite graph is 2-colorable.