# Problem Set 4

**Due: Tuesday, March 3.**

**Reading assignment**: 6.042 Fall 97 Notes for Lectures 6 and 7.

## Problem 1. [10 points]

Consider a robot which wanders on a plane. Our robot can move unit distance along any diagonal line (that is at an angle of $\pm 45$ degrees or $\pm 135$ degrees). If our robot starts at point $(0, 0)$, can it ever reach the point $(1, 1)$? Prove your answer.

## Problem 2. [20 points]

We are given two buckets, $A$ and $B$, a water hose, a receptacle, and a drain. The buckets and receptacle are initially empty. The buckets are labeled with their respectively capacities, positive integers $a$ and $b$. The receptacle can be used to store an unlimited amount of water, but has no measurement markings. Excess water can be dumped into the drain. Among the possible moves are:

1. fill a bucket from the hose,
2. pour from the receptacle to a bucket until the bucket is full or the receptacle is empty, whichever happens first,
3. empty a bucket to the drain,
4. empty a bucket to the receptacle,
5. pour from $A$ to $B$ until either $A$ is empty or $B$ is full, whichever happens first,
6. pour from $B$ to $A$ until either $B$ is empty or $A$ is full, whichever happens first.

**(a)** [**4 points**]
Model this scenario with a state machine. (What are the states? How does a state change in response to a move?)

**(b)** [**8 points**]
Prove that we can put $k \in \mathbb{N}$ gallons of water into the receptacle using the above operations iff $\gcd(a, b) \mid k$. *Hint*: Use the fact that if $a, b$ are positive integers then there exist integers $s, t$ such that $\gcd(a, b) = sa + tb$.

**(c)** [**8 points**]
Let's restrict the valid moves to numbers 1 and 4 only. Prove that there exists a number $n$, such that for all integer $k \geq n$, we can fill the receptacle with $k$ gallons of water iff $\gcd(a, b) \mid k$.

**Problem 3.** [**25 points**]

Consider the following game for two players. The players alternate moves. A move consists of a pair $(x, y)$ of natural numbers, subject to the constraint that no move may be $\geq$ to any previous move in both coordinates. The player who makes the $(0, 0)$ move is the loser.

At any point in the game, a valid move $(a, b)$ is said to be *bounded* iff both $a$ is bounded above by the $x$-coordinate of some previous move and $b$ is bounded above by the $y$-coordinate of some previous move. For example, if the previous moves have been $(1, 3), (3, 1), (2, 2)$, then there are ten possible bounded moves: $\{(1, 2), (2, 1), (1, 1), (0, 3), (0, 2), (0, 1), (0, 0), (1, 0), (2, 0), (3, 0)\}$. There is also an infinite number of possible unbounded moves at this point, namely, any move with one coordinate equal to 0 and the other larger than 3.

For any state $S$ of the game after the first move, let $X(S)$ be the minimum of the $x$-coordinates of the previous moves, $Y(S)$ be the minimum of the $y$-coordinates, and $B(S)$ be the number of possible bounded moves. Define the *measure*, $M(S)$, of the state to be the triple $(X(S), Y(S), B(S))$.

Define the following ordering on triples of natural numbers:

$$(x, y, k) < (x', y', k') \iff [(x < x') \vee (x = x' \wedge y < y') \vee (x = x' \wedge y = y' \wedge k < k')$$

**(a)** [**10 points**] Prove that any nonempty subset of $\mathbb{N}^3$ has a least element under the above ordering.

**(b)** [**10 points**] Conclude that the game always terminates, regardless of the players' strategy.

**(c)** [**5 points**]
Does any player have a winning strategy? Prove your answer.

**Problem 4.** [**45 points**]

The Stable Marriage Problem was described in lecture. In this problem we generalize slightly to handle "bachelors." The setup is that there is a finite number, not necessarily equal, of men and women; each man has a strict (no ties) ranking of *some* of the women according to his preference. The women on a man's list are said to be *acceptable* to him. Likewise each woman has a ranking of the men who are acceptable to her.

A *marriage set* is a bijection (matching) between a subset of the men and a subset of the women such that each man is matched to a woman he finds acceptable, and each woman is matched to a man she finds acceptable. In other words, a marriage set matches up men and women as "husband" and "wife" so that each man has exactly one wife, and each woman exactly one husband. Moreover, each wife is on her husband's list of ranked women, and each husband is on his wife's list of ranked men. Men and women not in the marriage set are called *bachelors*.

A husband in a marriage set is said to be *tempted* by a woman if (1) he prefers the woman to his wife, and (2) either the woman prefers him to her own husband, or the woman is a bachelor who finds him acceptable. Similarly, a wife may be tempted by a man. A marriage set is *stable* iff no husband is tempted by any woman and no wife is tempted by any man.

**(a)** [**5 points**] For all $n > 1$, describe a set of $n$ men and $n$ women for whom there is exactly one stable marriage set in which they are all married.

We can approach the Stable Marriage Problem with Bachelors using essentially the same protocol as in lecture (the only difference is the italicized phrase "*from a man she finds acceptable*" below):

### The Mating Algorithm

The mating ritual takes place over several days. The following events happen each day:

**Morning**: Each man who has not yet crossed all the women off his rank list, "proposes" to the woman he most prefers among those remaining on his list.

**Afternoon**: Each woman who receives at least one proposal *from a man she finds acceptable*, says to her favorite suitor, "Maybe.... Come back tomorrow." To the other suitors, she says, "Cross me off your list." We say that on this afternoon, the woman is *engaged* to the man to whom she said "Maybe ... " (Some women may not be engaged to anybody on a given afternoon.)

**Evening**: Each man who hears a "Cross me off ... " message, crosses the woman who gave him the message off his list.

**Termination condition**: The algorithm terminates on the first afternoon when every man who proposed to a woman that morning ends up engaged to her in the afternoon. Then all the engaged couples get married.

**(b)** [**5 points**] Model the problem as a state machine. State clearly what the states and state transitions are. What is the start state?

**(c)** [**8 points**] Use derived variables to prove termination. How long does the protocol take to terminate?

**(d)** [**7 points**] Prove that the marriage set at the end of the protocol is stable.

**(e)** [**5 points**] Suppose that no person is unacceptable to more than one person of the opposite sex. Show that there can be a situation in which every stable marriage set leaves two bachelors of each sex. *Hint*: Find a situation involving exactly two men and two women.

**(f)** [**8 points**] Suppose that the number of women and men is the same, and that no person is unacceptable to more than one person of the opposite sex. Show that the protocol leads to a stable marriage set where at most two men are bachelors.

**(g)** [**7 points**]

Let's modify the mating algorithm above by introducing the following change to the procedure for the Afternoon events. On each afternoon, all the women say "Maybe ... " as before, but only one woman says "Cross me off your list" and she says it to only one (but any one) among the suitors she would have said it to in the protocol above.

In particular, on each afternoon at most one "Cross me off your list" is spoken. Notice that now the state of the men is no longer uniquely determined by which afternoon it is. Briefly indicate whether this refined protocol always yields a stable final marriage set. When there is a final marriage set, is it necessarily the same as the one from the original protocol?