Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professor Albert R. Meyer

Handout 6
February 10, 1998

Problem Set 2

Due: In lecture, Thursday, February 19.

(2/11: Problems 5 and 7 corrected.)

Reading assignment: Sections 3.2–3 of Rosen. Finish reading *The Nuts and Bolts of Proofs*.

Problem 1. Prove by induction: For all $x, n \in \mathbb{N} : x + 1 | x^{2n+1} + 1$.

Problem 2. Prove that $1 \cdot 1! + 2 \cdot 2! + ... + n \cdot n! = (n+1)! - 1$ for all $n \in \mathbb{N}$.

Problem 3. For $n \in \mathbb{N}$, define numbers u_n by the equations:

$$u_0 = 2, u_1 = 3, u_{n+2} = 3u_{n+1} - 2u_n.$$

Guess a general formula for u_n and use mathematical induction to prove your answer.

Problem 4. A set of lines in the plane is said to be in *general position* when no two of the lines are parallel, and no three intersect at a common point. Let R_n be the number of regions into which n lines in general position divide the plane. For example, $R_0 = 1, R_1 = 2, R_2 = 4$. Explain, informally but clearly, why $R_{n+1} = R_n + (n+1)$. Find a simple formula for R_n and prove your result by induction.

Problem 5. Prove that any finite set of size $n > 0$ has exactly 2^{n-1} subsets of even size. For example, $\{1, 2\}$ has two subsets of even size: $\emptyset$ and $\{1, 2\}$. *Hint*: Let e be an element of a set S of size $n + 1$. An even size subset of S which does not contain e corresponds to an even size subset of $S - \{e\}$; an even size subset of S which does contain e corresponds to an odd size subset of $S - \{e\}$. So $\text{Even}(n+1) = \text{Even}(n) + \text{Odd}(n)$.

Problem 6. Pinpoint, and illustrate with a counterexample, *exactly* where the following proofs go awry.

(a) Claim 1: $(\forall \text{real } x)(\forall n \in \mathbb{N}) : x^n = 1$.
Proof by strong induction on n:
(Base case, $n = 0$) $x^n = x^0 = 1$ by definition of "zeroth power" of a real number.
(Strong Induction hypothesis) For all $k \leq n : x^k = 1$.
We must show that $x^{n+1} = 1$. By the strong inductive hypothesis, with $k = n$ and $k = 1$, we have $x^n = 1$ and $x^1 = 1$. So,

$$x^{n+1} = x^n x^1 = 1 \times 1 = 1.$$

■

(b) Claim 2: For any positive integers $p, n, x_1, x_2, \ldots, x_n$, if $p | x_1 x_2 \ldots x_n$, then $p | x_i$ for some $1 \leq i \leq n$.
Proof by induction on n.
(Base case, $n = 1$) When $n = 1$, we have $p | x_1$, therefore we can let $i = 1$ and conclude $p | x_i$.
Induction hypothesis: Now assuming the claim holds for n, we must prove it holds for $n + 1$.
So suppose $p | x_1 x_2 \ldots x_{n+1}$. If $p | x_1 \ldots x_n$, then by the inductive hypothesis we know that $p | x_i$ for some $1 \leq i \leq n$, and we have the desired i. Otherwise, $p | x_{n+1}$, so $i = n + 1$ will do. ■

Problem 7. Define *constant-free one-occurrence* Boolean formulas, *1BF*'s for short, inductively as follows:

- a variable is a 1BF.
- if F is a 1BF, then so is $\neg F$,
- if F and G are 1BF's with no variables in common, then $F \wedge G$ and $F \implies G$ are 1BF's.

A *Boolean assignment*, A, is a mapping from variables to truth values. The *complement* of assignment A is the assignment B such that $B(x) = \neg A(x)$ for all variables x in the domain of A.

Prove by induction that for all 1BF's, F, there is a Boolean assignment A for the variables of F such that F is true under assignment A and false under the complement of A.