A One-Sided Chebyshev Bound

We prove the following Lemma which was stated without proof in the notes for Fall '97, Lecture 24, §3.2

Lemma. For any random variable X and real number $c > 0$,

$$\Pr[X - \mathrm{E}[X] \geq c] \leq \frac{\mathrm{Var}[X]}{\mathrm{Var}[X] + c^2}.$$

Proof. Let $V ::= X - \mathrm{E}[X]$, so $\Pr[X - \mathrm{E}[X] \geq c] = \Pr[V \geq c]$. Also $\mathrm{E}[V] = 0$ and

$$\mathrm{Var}[X] = \mathrm{Var}[V] = \mathrm{E}[V^2].$$

Let $p ::= \Pr[V \geq c]$. We want an upper bound on p. If $p = 0$, the upper bound of the Lemma is immediate. Also, if $p = 1$, then $\mathrm{E}[V] \geq c > 0$, contradicting the fact that $\mathrm{E}[V] = 0$.

So we may assume $0 < p < 1$. Now the conditional expectations $\mathrm{E}[V \mid V \geq c]$ and $\mathrm{E}[V \mid V < c]$ will be finite, and we can apply a standard identity which holds for any V and c:

$$\mathrm{E}[V] = \mathrm{E}[V \mid V \geq c]\, p + \mathrm{E}[V \mid V < c]\,(1-p). \tag{1}$$

From the definition of conditional expectation, we have

$$\mathrm{E}[V \mid V \geq c] \geq c, \tag{2}$$

and from (1), (2), and the fact that $\mathrm{E}[V] = 0$, we obtain

$$0 \geq cp + \mathrm{E}[V \mid V < c]\,(1-p),$$

so

$$-\mathrm{E}[V \mid V < c] \geq \frac{cp}{1-p}. \tag{3}$$

Applying the standard identity again to V^2, we have

$$\mathrm{E}\left[V^2\right] = \mathrm{E}\left[V^2 \mid V \geq c\right] p + \mathrm{E}\left[V^2 \mid V < c\right](1-p), \tag{4}$$

but

$$\mathrm{E}\left[V^2 \mid V \geq c\right] \;\geq\; c^2, \tag{5}$$

and since the expectation of the square is at least the square of the expectation, we also have

$$\mathrm{E}\left[V^2 \mid V < c\right] \;\geq\; \mathrm{E}^2\left[V \mid V < c\right]. \tag{6}$$

Combining (4), (5), (6), and (3) gives

$$\mathrm{E}\left[V^2\right] \;\geq\; c^2 p + \left(\frac{cp}{1-p}\right)^2 (1-p),$$

and solving for p yields

$$p \leq \frac{\mathrm{E}\left[V^2\right]}{\mathrm{E}\left[V^2\right] + c^2} = \frac{\mathrm{Var}\left[V\right]}{\mathrm{Var}\left[V\right] + c^2}$$

as desired.

$\square$

Exercise: For each $\sigma > 0$, show that there is a random variable, V, with expectation zero and variance σ^2, for which the bound of the Lemma is tight. *Hint*: X need only take two values.