Massachusetts Institute of Technology | Handout 56
6.042J/18.062J: Mathematics for Computer Science | May 5, 1998, 11pm
Professor Albert R. Meyer

The Expectation of a Sum

We stated in Lecture 22 (4/30/98) that the expectation of the sum two random variables is the sum of their expectations. From this it's easy to prove by induction that the expectation of the sum of any *finite* number of random variables is the sum of their expectations. However, there are a number of applications where it's useful to know that this result actually generalizes to any *countable* number of random variables.

1 Countable Sum of Expectations

Theorem 1.1. *Let* $\mathbf{V} ::= (V_0, V_1, \ldots)$ *be a countable sequence of random variables on the same sample space such that*

$$\Pr\left[\sum_{i=0}^{\infty} |V_i| \ \textit{converges}\right] = 1.$$

Let V *be the random variable such that*

$$V = \sum_{i=0}^{\infty} V_i$$

on sample points where the sum converges, and $V = 0$ *otherwise. Then*

$$\mathrm{E}[V] = \sum_{i=0}^{\infty} \mathrm{E}[V_i].$$

Proof. Let r be a variable ranging over the values of V. Then by definition

$$\mathrm{E}[V] = \sum_{r} r \Pr[V = r]. \tag{1}$$

Notice that this is a countable sum, since there are at most as many values of r as points in the sample space.

Let $\mathbf{r}$ be a variable ranging over the infinite sequences of values of $V_0, V_1, \ldots$ at all sample points at which $V \neq 0$. Let i range over $\mathbb{N}$, and let r_i be the ith element of $\mathbf{r}$. So $\sum_i r_i$ is absolutely convergent by hypothesis.

We write $\mathbf{V} = \mathbf{r}$ for the event that $V_i = r_i$ for all i. So

$$\Pr[V = r] = \sum_{\substack{\mathbf{r}, \\ \sum_i r_i = r}} \Pr[\mathbf{V} = \mathbf{r}]. \tag{2}$$

The sum above is also countable, since there are again at most as many values of $\mathbf{r}$ as points in the sample space.

Combining (1) and (2), we have

$$\mathrm{E}[V] = \sum_{\mathbf{r}} \Big[\Big(\sum_i r_i\Big) \Pr[\mathbf{V} = \mathbf{r}]\Big]. \tag{3}$$

Let s range over the values of all of the random variables V_i. Since each V_i has only a countable number of values, the range of s is also countable.

Now, we can rearrange terms in absolutely convergent sums:

$$\begin{aligned}
&\sum_{\mathbf{r}} \Big[\Big(\sum_i r_i\Big) \Pr[\mathbf{V} = \mathbf{r}]\Big] \\
&\quad = \sum_{\mathbf{r}} \Big[\sum_i r_i \Pr[\mathbf{V} = \mathbf{r}]\Big] \\
&\quad = \sum_i \sum_{\mathbf{r}} r_i \Pr[\mathbf{V} = \mathbf{r}] \\
&\quad = \sum_i \sum_s \sum_{\substack{\mathbf{r}, \\ r_i = s}} s \Pr[\mathbf{V} = \mathbf{r}] \\
&\quad = \sum_i \sum_s \Big[s \sum_{\substack{\mathbf{r}, \\ r_i = s}} \Pr[\mathbf{V} = \mathbf{r}]\Big] \\
&\quad = \sum_i \sum_s s \Pr[V_i = s] \\
&\quad = \sum_i \mathrm{E}[V_i]
\end{aligned}$$

$\square$

2 Gambler's Ruin: The Fair Case

An elegant application of a countable sum of expectations arises from the "Gambler's Ruin" problem.

A gambler starts with an initial stake of n dollars. He repeatedly flips a coin, betting one dollar on heads each time. We assume the results of different flips are independent and that the coin is fair: the probability of heads is $1/2$. When heads comes up, the gambler wins one dollar; when tails comes up, he loses one dollar. The game ends when either the gambler *loses* by going broke, or *wins* by reaching a goal amount of $T \geq n$ dollars. We are interested in the probability, w, that the gambler wins. This is called the Gambler's Ruin problem.

Of course, we are also interested in the probabilities of the other two possibilities: that the gambler loses, or that the gambler keeps flipping forever without either winning or losing. We will show later that the probability of playing forever is zero. This means that the probability that the gambler loses is $1 - \Pr[\text{gambler wins}] = 1 - w$.

We can visualize the Gambler's Ruin problem as a random walk illustrated in Figure 1. At each step, if the gambler wins, the path goes up by 1—this happens with probability $1/2$; otherwise it goes down by 1—also with probability $1/2$. If the path touches either the top barrier at T, or the bottom barrier at 0, the game ends.

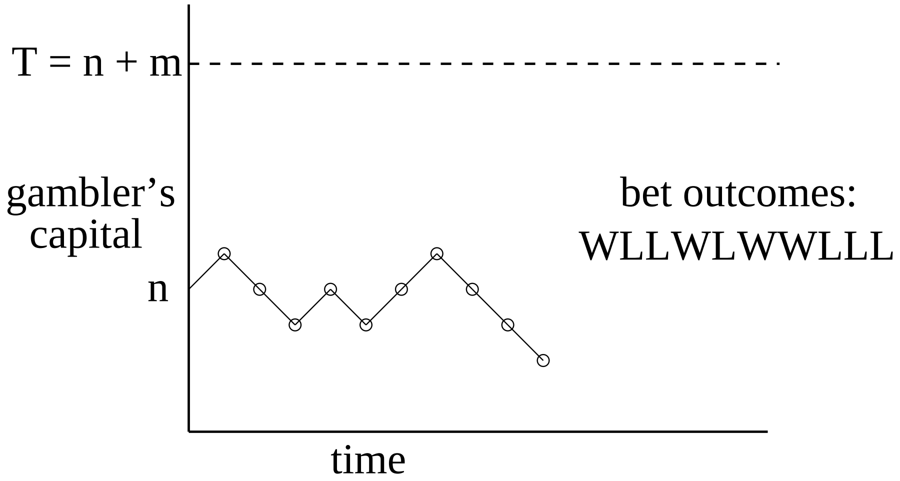

Figure 1: This is a graph of the gambler's capital versus time for one possible sequence of bet outcomes. At each time step, the graph goes up with probability $1/2$ and down with probability $1/2$. The gambler starts with n and continues betting until the graph reaches either 0 or T.

We can solve for w by considering the random variable G equal to the gambler's dollar gain. The gain is unspecified in the case that the game goes on forever, but since this happens with probability zero, it doesn't matter how G behaves in this case; say it equals 0.

Now $G = T - n$ when the gambler wins the game, and $G = -n$ if the gambler loses (goes broke). Therefore,

$$\begin{aligned}
\mathrm{E}[G] &= \Pr[\text{Gambler wins}](T - n) - \Pr[\text{Gambler loses}]\, n \\
&= w(T - n) - (1 - w)n \\
&= wT - n,
\end{aligned}$$

and so

$$w = \frac{\mathrm{E}[G] + n}{T}. \tag{4}$$

How to find $\mathrm{E}[G]$? Here's the trick: let G_i be the amount the gambler gains on the ith flip: $G_i = 1$ if the gambler wins the flip, $G_i = -1$ if the gambler loses the flip, and $G_i = 0$ if the game has ended before the ith flip. Since the coin is fair, $\mathrm{E}[G_i] = 0$. Now

$$G = \sum_i G_i,$$

except for the probability zero event that the game runs forever. Moreover, $\sum_i |G_i|$ converges with probability one because $G_i = 0$ whenever i exceeds the length of the game. So the conditions for our Theorem on a countable of sum expectations are satisfied, and

$$\mathrm{E}[G] = \sum_{i=1}^{\infty} \mathrm{E}[G_i] = \sum_{i=1}^{\infty} 0 = 0.$$

So we conclude from (4),

$$\Pr[\text{Gambler wins}] = w = \frac{n}{T}.$$

Example 1. *Suppose we have \$100 and we start flipping a fair coin, betting \$1 with the aim of winning \$100. Then the probability of reaching the \$200 goal is* $100/200 = 1/2$*—the same as the probability of going bankrupt. In general, if* $T = 2n$*, then the probability of doubling your money or losing all your money is the same. This is about what we would expect.*

Example 2. *Suppose we have \$500 and we start flipping a fair coin, betting \$1 with the aim of winning \$100. So* $n = 500, T = 600$*, and* $\Pr[\mathit{win}] = 500/600 = 5/6$*.*

Example 3. *Suppose Albert starts with \$100, and Luca starts with \$10. They flip a fair coin, and every time a head appears, Albert wins \$1 from Luca, and vice versa for tails. They play this game until one person goes bankrupt. What is the probability of Albert winning?*

This problem is identical to the Gambler's Ruin problem with $n = 100$ *and* $T = 100 + 10 = 110$*. The probability of Albert winning is* $100/110 = 10/11$*, namely, the ratio of his wealth to the combined wealth. Luca's chances of winning are* $1/11$*.*

Although Albert will win most of the time, the game still has expected value zero *for him. When Albert wins, he wins only \$10. When he loses, he loses big—\$100.*

Another intuitive idea is confirmed by this analysis: the larger the gambler's initial stake, the larger the probability that he will win a fixed amount.

Example 4. *If the gambler started with one million dollars instead of 500, but aimed to win the same 100 dollars as in the Example 2, the probability of winning would increase to* $1M/(1M + 100) > .9999$.

We'll see in a later lecture that Gambler's Ruin behavior is very different when the coin is even slightly unfair.

We still have to show that $\Pr[\text{gambler plays forever}] = 0$.

Exercise: Let N_k be the event that the game lasts for at least k coin tosses.

(a) Argue that

$$\Pr[\text{gambler plays forever}] \leq N_{k+t} \leq N_k \left(1 - (1/2)^t\right).$$

(b) Conclude that $\Pr[\text{gambler plays forever}] = 0$.