Probabilities in a Dice Game

In Lecture 19 on 4/8/98 we described a simple dice game: two players take turns rolling two dice and whoever rolls an eleven first wins. It's pretty clear that the first player has an advantage, since he has the first chance to win. How much of an advantage? Here's a simple analysis.

1 The Probability that the First Player Wins

The first player has a probability of 1/18 of winning immediately on his first turn—even before the second player has had a turn—because two of the 36 outcomes of rolling a pair of dice yield a sum of eleven. So if we let E be the event that the first player rolls eleven on the first roll, we have

$$\Pr(E) = 1/18.$$

Let W be the event that the first player wins. We want to find $\Pr(W)$.

The first player wins with probability 1/18th on his first roll. With probability 17/18 he does not win on his first roll, in which case his probability of winning is, by definition, $\Pr(W|\overline{E})$. In other words, we have

$$\Pr(W) \quad = \quad \frac{1}{18} + \Pr(W|\overline{E}) \cdot \frac{17}{18}. \tag{1}$$

This leaves us the problem of finding $\Pr(W|\overline{E})$.

Instead of considering the case when the first player wins later than his first roll, let's consider the case when the second player wins. Now we can argue from symmetry: once the first player has failed to roll an eleven on his first roll, the situation for the second player is exactly the same as it was on the previous roll for the first player. That is, the probability that the second player wins, given that the first player did not roll an eleven on his first term, must also be $\Pr(W)$:

$$\Pr(W) \quad = \quad \Pr(\text{second player wins} \mid \overline{E}). \tag{2}$$

Now let's make a simplifying assumption about this game which we will justify shortly: assume that *somebody must always win*, that is, ties are impossible. This means that the second player wins iff the first player does *not* win, so

$$\Pr(\text{second player wins} \mid \overline{E}) \quad = \quad \Pr(\overline{W} \mid \overline{E}). \tag{3}$$

Then from equations (2) and (3),

$$\Pr(W) \;=\; \Pr(\overline{W} \mid \overline{E}),$$

and since $\Pr(\overline{W} \mid \overline{E}) = 1 - \Pr(W \mid \overline{E})$, we conclude

$$\Pr(W \mid \overline{E}) = 1 - \Pr(W). \tag{4}$$

Finally, from equations (1) and (4) we have

$$\Pr(W) = \frac{1}{18} + (1 - \Pr(W)) \cdot \frac{17}{18}.$$

Solving for $\Pr(W)$ we conclude

$$\Pr(W) = \frac{18}{35} \approx 0.514.$$

In other words, the first player has about a 1.4% advantage.

2 The Possibility of a Tie

To complete the analysis, we have to justify the assumption that ties are impossible. Now that's not really true according to the rules of the game: if both players always roll non-elevens, the game could go on forever without a winner and would be a tie. But we used the "no ties" assumption above only to justify equation (3), and we can justify it assuming that a tie, rather than being impossible, merely has probability zero. That's because at any point in the game, the first player does *not* win iff the second player *wins or ties*, so

$$\begin{aligned}
\Pr(\overline{W} \mid \overline{E}) &= \Pr(\text{second player wins or ties} \mid \overline{E}) \\
&= \Pr(\text{second player wins} \mid \overline{E}) + \Pr(\{\text{a tie}\} \mid \overline{E}) \\
&= \Pr(\text{second player wins} \mid \overline{E}) + 0 \\
&= \Pr(\text{second player wins} \mid \overline{E}),
\end{aligned}$$

proving equation (3).

So, we need to establish that the probability of a tie is zero. To do this, we will, finally, have to describe explicitly the probability space by which we model the game.

3 The Probability Space

In this game all that matters about a dice roll is whether or not it sums to eleven, so it seems natural to model a sequence of dice rolls by a binary sequence, where the ith bit is 1 iff the ith roll was eleven. There is a winner in a particular game iff the game consisted of a sequence of non-eleven rolls and ended with an eleven; this corresponds to a sample point consisting a sequence of $n \geq 0$ zeroes followed by a one. We also expect the probability of eleven on any give roll remains 1/18 no matter what happens on the other rolls—the outcomes of different rolls are independent. So it's natural to take the probability of such a sequence to be $(17/18)^n(1/18)$.

In addition to these winning sample points, there is one more sample point corresponding to a tie, namely, the infinite sequence of 0's. There's no reason to include any other binary sequences—finite or infinite—in our sample space, since they do not correspond to game outcomes.

Now

$$\begin{aligned}
\Pr(\text{someone wins}) &= \sum_{m\geq 1} \Pr(\text{someone wins on the } m\text{th roll}) \\
&= \sum_{m\geq 1} (\frac{17}{18})^{m-1}\frac{1}{18} \\
&= \frac{1}{18}\sum_{n\geq 0}(\frac{17}{18})^n \\
&= \frac{1}{18}\cdot\frac{1}{1-\frac{17}{18}} = 1,
\end{aligned}$$

and

$$\Pr(\text{a tie}) = 1 - \Pr(\text{someone wins}) = 1 - 1 = 0.$$

This completes the definition of the probability space on which the analysis above implicitly rested.

Now that we have the space defined explicitly, we can verify the informal argument "by symmetry" above with a rigorous proof:

Exercise: Prove that

$$\Pr(W) = \Pr(\text{second player wins} \mid \overline{E}),$$

by describing a suitable bijection between the event $\overline{E}$ and the whole probability space.