Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Notes on Book Stacking

We consider a pile of unit-length books, stacked one on another and placed on a table. The center of mass of the whole stack must lie over the table or it would pivot over the edge. In fact, if the center of mass of the stack on top of the ith book was not positioned on some vertical line through the ith book, then the stack would pivot over the edge of that book, as shown in Figure 1. On the other hand, the stack will be physically *stable* providing that for every i, the center of mass of the top $i-1$ books lies somewhere vertically above book i, and the center of mass of the whole stack lies vertically over the table.

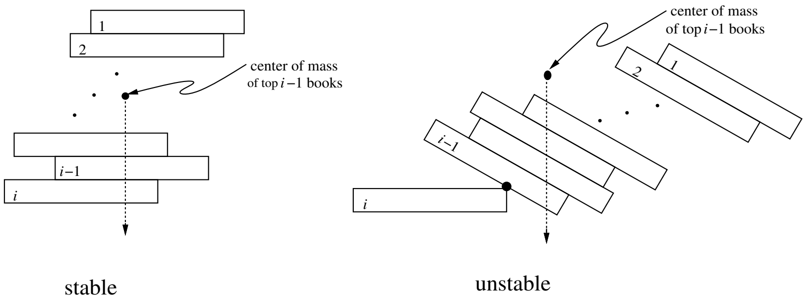

Figure 1

Define the *overhang* of a stack of books to be the distance from its center of mass to the end of the book furthest to the right of the stack. When the stack is placed so its center of mass is at the edge of the table, which is as far to the right as the stack can be placed without tipping, the overhang of the stack measures exactly how far the rightmost book extends past the table edge, as shown in Figure 2. So we want a formula for the maximum possible overhang, B_n, achievable with a stack of n books.

For example, with one book, the center of mass is 1/2 a book length from the righthand edge of the book, so the overhang is 1/2. That is,

$$B_1 = \frac{1}{2}.$$

A "stack" of one book is stable as long as the center of mass of the book is over the table. So by placing the center of mass at the edge of the table, we can have the righthand edge stick out by the overhang, as shown in Figure 3.

Now suppose we have a stack of $n+1$ books with maximum overhang. If the overhang of the stack of n books on top of the the bottom book was not maximum, we could replace it with a stack of n books with larger overhang, positioning the new stack so its center of mass was at the same position as the original stack. In other words, the maximum overhang,

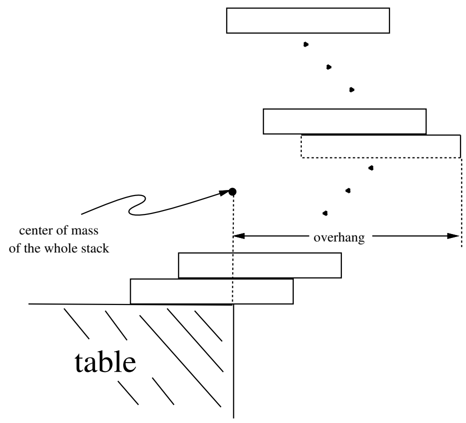

Figure 2

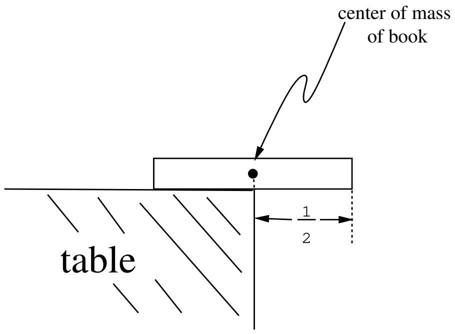

Figure 3

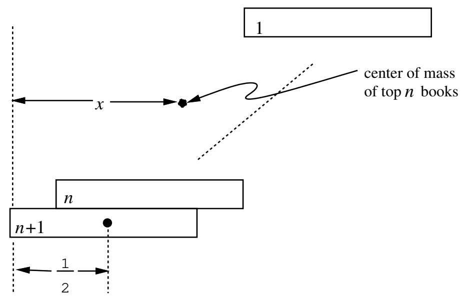

Figure 4

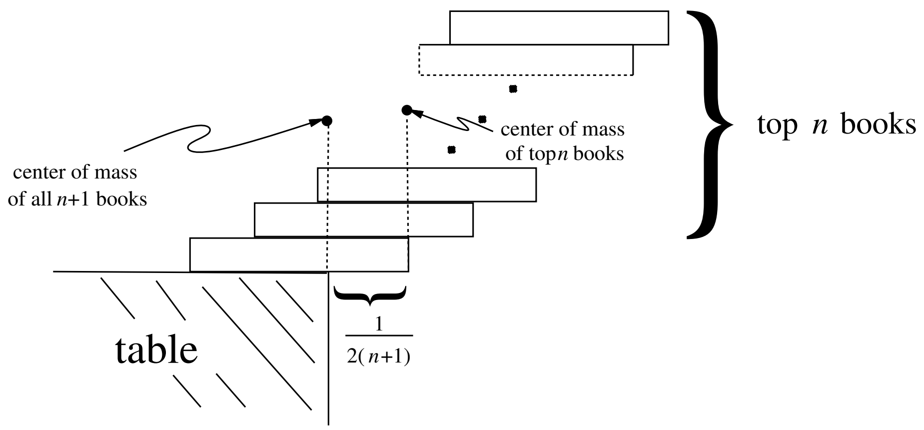

Figure 5

B_{n+1}, of the stack of $n+1$ books is obtained by placing a stable stack of n books with maximum overhang, B_n, somewhere on top of the bottom book.

Suppose we place the stable stack of n books on the $(n+1)$st bottom book so that the stack's center of mass is distance x to the right of the left edge of the bottom book, as shown in Figure 4. The center of mass of the top n books must lie above the bottom book for the stack of all $n+1$ books to be stable, so we must have $0 \leq x \leq 1$.

Now the center of mass of the stack of $n+1$ books is distance $(1 \cdot 1/2 + nx)/(n+1)$ to the right of the left edge of the bottom book, so the overhang of the stack of $n+1$ books has increased by $x - (1 \cdot 1/2 + nx)/(n+1) = (2x-1)/2(n+1)$. This increase is maximized when x takes its maximum value 1. So the maximum increase in overhang is $1/2(n+1)$, and occurs when the center of mass of the top n books is placed exactly at the righthand edge of the bottom book, as shown in Figure 5.

So,

$$B_{n+1} = B_n + \frac{1}{2(n+1)}$$

and we have

$$
\begin{aligned}
B_{n+1} &= \frac{1}{2(n+1)} + \frac{1}{2n} + \frac{1}{2(n-1)} + \cdots + \frac{1}{2\cdot 2} + \frac{1}{2\cdot 1} \\
&= \frac{1}{2}\sum_{i=0}^{n} \frac{1}{(n+1)-i} \\
&= \frac{1}{2}\sum_{i=1}^{n+1} \frac{1}{i}.
\end{aligned}
$$

Let H_n be the value of $\sum_{i=1}^{n}(1/i)$, so $B_n = H_n/2$. H_n is called the nth *Harmonic number*. In lecture we used the integral method to conclude that $H_n \sim \ln n$. In particular, H_n grows slowly, but unboundedly with n. So we can get books to overhang any distance past the edge of the table by piling them high enough!