Conflict Final Exam

- This exam is closed book. Handout 60 with results and formulas is attached for reference.
- This exam contains ten (10) problems totaling 200 points. Problems are labelled with point values.
- Put your name on the top of **every** page.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem.
- GOOD LUCK!

Your Name:

Circle the name of your TA:

Alex Duran   Stas Jarecki   Al Lee   Edmond Lee   Mark Roh   Tara Schenckel

Recitation Time:

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	25		
2	20		
3	25		
4	10		
5	25		
6	15		
7	20		
8	25		
9	15		
10	20		
Total	200		

Problem 1. [25 points] Circle True/False.

T F Adding k edges to an undirected graph increases its chromatic number (the number of colors needed to color the vertices) by at most k.

T F If an undirected graph has the degree of every vertex divisible by a prime p, then p must evenly divide the number of edges.

T F Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is a surjection, then both f and g are surjections.

T F Assume each Boolean variable is randomly assigned true or false with equal likelihood and the choices are independent. Then $\Pr[(p_1 \vee p_2 \vee \bar{p_3}) \wedge (\bar{q} \vee r)$ is true$] \leq 2/3$.

T F $\forall x : [P(x) \implies \forall y : Q(y)]$ implies $\exists x : \neg Q(x) \implies \neg \exists x : P(x)$.

T F The relation $\{(A, B) \mid$ there is a surjection from A to $B\}$ is transitive.

T F The set of infinite subsequences of the decimal representation of π is uncountable.

T F $\log(n!) = n^{f(n)}$ for some $f = \Theta(\log n)$.

Problem 2. [20 points]

For each of the following statements, indicate whether it is true or false. If it is true, sketch a proof; if it is false, describe a counterexample.

(a) [5 points] All nine-vertex trees with exactly two eight-vertex paths are isomorphic.

(b) [7 points] In the polynomial expansion of $(x-1)^{120}$, every coefficient occurs exactly twice.

(c) [8 points] The number of sequences of eleven letters which can be made from the letters in MISSISSIPPI is not divisible by four.

Problem 3. [25 points] Let $T : \mathbb{N} \to \mathbb{N}$ be defined by the following recurrence. Derive a closed form expression for an $f(n)$ such that $T(n) = \Theta(f(n))$.

$$T(n) = \begin{cases} 8 & \text{if } n \leq 5, \\ T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\left\lfloor \frac{3n}{4} \right\rfloor\right) + 8n & \text{if } n > 5. \end{cases}$$

Problem 4. [10 points] Write a closed form expression for $\Pi_{k=1}^{n} 2^{2^k}$.

Problem 5. [25 points] Short answers. No need to be rigorous, though partial credit may be given based on the derivation.

(a) [10 points]
We'll say an integer $n \geq 2$ is "good" providing that in any set of 401 integers, there are three integers, i, j, and k, such that the pairwise differences, $j-k, j-l, k-l$ are multiples of n. There is a bound $k \in \mathbb{N}$ such that n is good iff $2 \leq n \leq k$. What is k?

(b) [5 points] How many paths of length 7 are there in an n vertex complete simple graph (remember, there are no self-loops)?

(c) [5 points] The space shuttle runs 3 identical computers so that if one goofs, the other two will outvote it with the right answer. If each computer goofs independently with probability g, what is the probability that the right answer will prevail?

(d) [5 points] Suppose I flip three fair coins and then roll as many 6-sided dice as there are coins showing heads. What is the expected sum of the dice?

Problem 6. [15 points]

Prove that there are 16,800 permutations of the letters $\{a, b, c, d, e, f, g, h\}$ such that neither the letters $\{a, b, c\}$ nor the letters $\{d, e\}$ occur in order. For example, $fbgadceh$ is not allowed because d occurs before e, and $fagbecdh$ is not allowed because a occurs before b and b occurs before c, but $fbgaedch$ is OK.

Problem 7. [20 points]

A high blood pressure test is known to be 90% accurate in detecting high blood pressure (that is, 10% false negatives), whereas only 5% of people without high blood pressure will falsely test positive. Suppose 30% of the population has high blood pressure. Given that a person picked at random tests positive, what is the conditional probability that this personhas high blood pressure?

Problem 8. [25 points]

An urn contains red, white and blue colored balls. The fraction of red, white, and blue balls is r, w, b, respectively (so $r + w + b = 1$). A gambler repeatedly pulls a ball at random from the urn. That is, at any point during succesive pulls, each ball in the urn is equally likely to be the next one pulled. He *replaces* each ball into the urn before making the next pull. He keeps pulling balls until he pulls two red ball *in a row.*

(a) [5 points] Let N_n be the event that at least n balls have been pulled and the nth ball is *not* red; let $p_n = \Pr[N_n]$. Write an expression in terms of p_n for the probability that the Gambler stops pulling after exactly $n + 2$ balls.

(b) [10 points] Write a linear homogeneous recurrence equation for p_{n+2} in terms of p_{n+1} and p_n.

(c) [10 points] Suppose $r = 2/3$. Write a closed form expression for p_n.

(d) [5 points] *Outline in a few words* how the previous result could be used to calculate the expected number of balls pulled by the Gambler. You need not actually carry out the calculation.

Problem 9. [15 points] There is a very unreliable test for contaminated water: it is as likely to be correct as not when an uncontaminated sample is tested. It is slightly *less* likely to be correct on a contaminated sample. Is there any way a reliable contamination test can be developed based on the unreliable one? Briefly explain why not, or explain how to do it.

Problem 10. [20 points]

The prosecution at OJ's trial is attempting to tie OJ to the crime by a new analysis of *blood thickness*. Blood thickness is expressed as a natural number. The average blood thickness over all humans is 100. OJ has very thick blood: its thickness is 150.

(a) [10 points]
Knowing only the average thickness, what bounds (upper and lower) can you prove on the fraction of humans that can have blood thickness 150?

(b) [10 points]
At the trial, the prosecution reveals that the standard deviation of blood thickness is 5. It is further revealed that the blood found at the scene of the crime has thickness 150. Using Chebyshev's theorem, what can you say about the fraction of humans that might have committed the crime?