Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professor Albert R. Meyer

Handout 60
May 20, 1998

Notes for Final Exam

Summation

$$
\begin{aligned}
\sum_{i=0}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{i=0}^{n} x^i &= \frac{1-x^{n+1}}{1-x} \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \dots \\
\sum_{i=1}^{n} ix^i &= \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2} \\
\frac{x}{(1-x)^2} &= x + 2x^2 + 3x^3 + 4x^4 + \dots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
-\ln(1-x) &= \int \frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots
\end{aligned}
$$

Master Theorem

For $T, f : \mathbb{N} \to \mathbb{N}$, and constants $a > 0, b > 1$, if

$$T(n) = aT(\lceil n/b \rceil) + f(n),$$

then

$$
T(n) = \begin{cases} \Theta(n^\beta) & \text{if } f(n) = O(n^{\beta-\epsilon}) \text{ for some } \epsilon > 0, \\ \Theta(n^\beta \log n) & \text{if } f(n) = \Theta(n^\beta), \\ \Theta(f(n)) & \text{if } af(n/b) \le cf(n) \text{ for some } c < 1 \text{ and all large } n. \end{cases}
$$

where $\beta ::= \log_b a$.

Linear Recurrences

A homogeneous linear recurrence of order d,

$$f(n) = a_1 f(n-1) + a_2 f(n-2) + \dots + a_d f(n-d),$$

has characteristic polynomial:

$$p(\alpha) ::= \alpha^d - a_1\alpha^{d-1} - a_2\alpha^{d-2} - \dots - a_{d-1}\alpha - a_d,$$

and characteristic equation $p(\alpha) = 0$. If r is a root of p with multiplicity $m > 0$, then all of

$$r^n,\ nr^n,\ n^2r^n,\ \dots,\ n^{m-1}r^n$$

are solutions to the recurrence. Every solution to the equation is a linear combination of the d terms obtained in this way from the characteristic equation.

Stirling's Formula

$$n! = \sqrt{2\pi n}(\frac{n}{e})^n e^{a_n}$$

where $1/(12n+1) \leq a_n \leq 1/12n$.

Counting

Given a set of n elements:

- number of r-permutations without replacement:

$$P(n,r) ::= n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

- r-combination without replacement:

$$\binom{n}{r} ::= \frac{n!}{(n-r)!\; r!}$$

- r-permutation with replacement (r distinct balls, n distinct bins): n^r
- r-combination with replacement (r identical balls, n distinct bins):

$$\binom{n+r-1}{r}$$

- number of permutations with repetition:

$$\binom{n}{r_1, r_2, \ldots, r_k} ::= \frac{n!}{r_1!\; r_2! \ldots r_k!}$$

  where $r_1 + r_2 + \cdots + r_k = n$.

Multinomial Theorem

$$(x_1 + x_2 + \ldots + x_k)^n = \sum_{r_1+r_2+\ldots+r_k=n} \binom{n}{r_1, r_2, \ldots, r_k} x^{r_1} x^{r_2} \ldots x^{r_k}$$

Inclusion-Exclusion

$$|A_1 \cup \ldots \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{S \subseteq \{1,\ldots,n\}, |S|=k} |\bigcap_{i \in S} A_i| \right)$$

Generating Functions

Let $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ and $B(x) = b_0 + b_1 x + \ldots$ be ordinary generating functions for the number of ways to choose items from disjoint sets $\mathcal{A}$ and $\mathcal{B}$. The generating function for the number of ways to choose objects from the union $\mathcal{A} \cup \mathcal{B}$ is the product $A(x) \cdot B(x)$. The coefficient of x^n in the product is

$$c_n ::= a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots a_n b_0.$$

A random variable R taking nonnegative integer values has Probability Generating Function:

$$\begin{aligned}
G_R(x) &::= \sum_{i=0}^{\infty} \Pr[R = i]\, x^i, \\
\mathrm{E}[R] &= G_R'(1), \\
G_{R_1+R_2} &= G_{R_1} \cdot G_{R_2} \quad \text{for independent } R_1, R_2.
\end{aligned}$$

Probability

Events

$$\begin{aligned}
\Pr\left[\bigcup_{n\in\mathbb{N}} A_n\right] &= \sum_{n\in\mathbb{N}} \Pr[A_n] \text{ for pairwise disjoint } A_n \\
\Pr\left[\overline{B}\right] &= 1 - \Pr[B], \\
\Pr[A \cup B] &= \Pr[A] + \Pr[B] - \Pr[A \cap B] \\
\Pr[A \cup B] &\le \Pr[A] + \Pr[B] \quad \text{[Boole's inequality]} \\
\Pr[A \mid B] &= \frac{\Pr[A \cap B]}{\Pr[B]} \\
\Pr[A \mid B] &= \frac{\Pr[B \mid A] \Pr[A]}{\Pr[B]} \quad \text{[Bayes Rule]}
\end{aligned}$$

Expectation

$$\begin{aligned}
\mathrm{E}[R] &::= \sum_{r\in\mathbb{R}} r \cdot \Pr[R = r] \\
\mathrm{E}\left[\sum_{n\in\mathbb{N}} R_n\right] &= \sum_{n\in\mathbb{N}} \mathrm{E}[R_n] \quad \text{for absolutely convergent} \sum_{n=1}^{\infty} R_n \\
\mathrm{E}[R_1 R_2] &= \mathrm{E}[R_1]\,\mathrm{E}[R_2] \quad \text{for independent } R_1, R_2
\end{aligned}$$

Variance

$$\begin{aligned}
\mathrm{Var}[R] &::= \mathrm{E}\left[(R - \mathrm{E}[R])^2\right] = \mathrm{E}\left[R^2\right] - \mathrm{E}^2[R], \\
\sigma_R &::= \sqrt{\mathrm{Var}[R]} \quad [\sigma_R ::= \text{the } \textit{Standard Deviation} \text{ of } R], \\
\mathrm{Var}[aR + b] &= a^2 \mathrm{Var}[R], \text{ for } a, b \in \mathbb{R}, \\
\mathrm{Var}[R_1 + \cdots + R_n] &= \mathrm{Var}[R_1] + \cdots + \mathrm{Var}[R_n] \quad \text{for pairwise independent } R_1, \ldots, R_n.
\end{aligned}$$

Binomial Distribution

$$
\begin{aligned}
f_{n,p}(k) &::= \binom{n}{k} p^k (1-p)^{n-k} \\
F_{n,p}(k) &::= \sum_{i=0}^{k} f_{n,p}(i) \\
f_{n,p}(\alpha n) &= 2^{\left(\alpha \log_2\left(\frac{p}{\alpha}\right) + (1-\alpha)\log_2\left(\frac{1-p}{1-\alpha}\right)\right)\cdot n} \overbrace{e^{a_n - a_{\alpha n} - a_{n-\alpha n}}}^{\sim 1} \frac{1}{\sqrt{2\pi\alpha(1-\alpha)n}} \\
F_{n,p}(\alpha' n) &\leq \frac{1-\alpha'}{1-\alpha'/p} f_{n,p}(\alpha' n)
\end{aligned}
$$

where $0 < \alpha < 1, 0 < \alpha' < p$, and a_n from Stirling's formula.

Wald's Theorem

Let Q be a non-negative integer valued random variable such that $\mathrm{E}[Q]$ is finite, and let $R_1, R_2, \ldots$ be a countable sequence of random variables. Let T be the sum $R_1 + R_2 + \cdots + R_Q$. Suppose that, conditioned on being included in the sum, every variable R_k has the same distribution as R_1, and also suppose that $\mathrm{E}[|R_1|]$ is finite. Then, $\mathrm{E}[T] = \mathrm{E}[Q] \cdot \mathrm{E}[R_1]$.

Markov's Theorem

If $R : S \to \mathbb{R}^+$ is a random variable, then for all $x \geq 0$,

$$
\Pr[R \geq x] \leq \frac{\mathrm{E}[R]}{x}.
$$

The Chebychev Bound

For any random variable R,

$$
\begin{aligned}
\Pr[|R - \mathrm{E}[R]| \geq x] &\leq \frac{\mathrm{Var}[R]}{x^2}, \\
\Pr[|R - \mathrm{E}[R]| \geq x\sigma_R] &\leq \frac{1}{x^2}, \\
\Pr[R - \mathrm{E}[R] \geq x] &\leq \frac{\mathrm{Var}[R]}{\mathrm{Var}[R] + x^2}. \quad \text{[One-sided version]}
\end{aligned}
$$

Chernoff Bound

Let T, be a sum of n independent 0-1-valued random variables. Then for all $c > 1$

$$
\begin{aligned}
\Pr[T \geq e\mathrm{E}[T]] &\leq e^{-\mathrm{E}[T]}, \\
\Pr[T \geq c\mathrm{E}[T]] &\leq e^{-\alpha c \mathrm{E}[T]},
\end{aligned}
$$

where $e \approx 2.72$ is the base of the natural logarithms and $\alpha ::= \ln c + 1/c - 1$.