Quiz

- This quiz is closed book. One two-sided crib sheet is allowed.
- Put your name on the top of **every** page—**these pages will be separated for grading.**
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. **Be neat**.
- This quiz ends at 9 P.M. It contains six (6) problems totaling 100 points. Each problem and problem part is labelled with the number of points it is worth.
- GOOD LUCK!

Your Name: ______________________________

Circle the name of your TA:
Alex Duran   Stas Jarecki   Al Lee   Edmond Lee   Mark Roh   Tara Schenckel

Write the hour at when your Friday recitation begins ______

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade
1	25	
2	20	
3	15	
4	15	
5	10	
6	15	
Total	100	

Problem 1. [25 pts]

Let L_n be the $n+1$-vertex simple "line" graph consisting of a single simple path of length n. For example, L_4 is shown in the figure below:

The line graph L_4

Let's say a simple graph has "two ends" if it has exactly two vertices of degree one, and all its other vertices have degree two. In particular, for $n \geq 1$, the graph L_n has two ends.

a) [3 pts] Draw a diagram of the smallest simple graph with two ends which is not isomorphic to any line graph. *Hint*: See part (d).

b) [8 pts] Explain briefly, but clearly, what is wrong with the following argument:

False theorem: Every simple graph with two ends is isomorphic to L_n for some $n \geq 1$.

False proof: We prove by induction on the number, $n \geq 1$, of edges in a simple graph, that every two-ended graph with n edges is isomorphic to L_n.

(Base case $n = 1$): A simple graph with one edge can only consist of the two vertices connected by that edge and some number of vertices not attached to any edge, i.e., vertices of degree zero. Since a two-ended graph cannot have vertices of degree zero, the only two-ended graph with one edge must consist solely of the two vertices connected by the edge, which makes it isomorphic to L_1.

(Induction case): Assume that $n \geq 1$ and every two-ended graph with n edges is isomorphic to L_n. Now let G_n be any two-ended graph with $n \geq 1$ edges. By hypothesis, G_n is isomorphic to L_n. Suppose an edge is added to G_n to form a two-ended graph G_{n+1}. Since G_n is isomorphic to L_n, it consists of a simple path of length n. The only way to add an edge to the path and preserve two-endedness is to have that edge go from one end—that is, one of the degree-one vertices—to a new vertex, lengthening the path by one. That is, the resulting $n+1$-edge graph must be a simple path of length $n+1$, so it is isomorphic to L_{n+1}. Q.E.D.

[Place your explanation on the next page.]

c) [7 pts] Here is another argument for the same False Theorem as in part (b). Explain exactly where it goes wrong.

False proof 2: Same induction hypothesis and base case as in part (b).

(Induction case): For any $n \geq 1$, let G_{n+1} be any two-ended graph with $n+1$ edges. Let G_n be the graph which results from removing one of the degree-one vertices v of G_{n+1} and the edge $\{v, w\}$ attached to it. So G_n no longer has the vertex, v, of degree one. But the degree of w is one less in G_n than it was in G_{n+1}, so G_n still has two vertices of degree one and one fewer vertex of degree two. Therefore G_n is also two-ended. By induction G_n consists of a simple path of length n. But G_{n+1} is obtained by attaching an edge from one end of the path to a vertex v not on the path, thereby lengthening the path by one. So G_{n+1} is isomorphic to a simple path of length $n + 1$; that is, it is isomorphic to L_{n+1}. Q.E.D.

d) [7 pts] Describe how to make a small revision to one of the false proofs above so that it becomes a correct proof of the theorem "Every *connected* simple graph with two ends is isomorphic to L_n for some $n \geq 1$."

Problem 2. [20 pts]

Consider the following simple graphs:

$$K_8 \quad C_8 \quad K_{4,4} \quad W_7 \quad Q_3 \quad L_7$$

(If you've forgotten how these graphs are defined, you can find the definitions in the Appendix of this quiz.)

a) [6 pts] What is the minimum number of colors (chromatic number) needed to vertex-color each of these graphs so that adjacent vertices have different colors?

K_8 ____ C_8 ____ $K_{4,4}$ ____ W_7 ____ Q_3 ____ L_7 ____

b) [8 pts] Let $v_1, v_2, \ldots$ be variables ranging over vertices and $E(v_1, v_2)$ mean that v_1 and v_2 are connected by an (undirected) edge. For each of the following logical statements, indicate exactly which of the six graphs above satisfies the statement.

i) $\exists v_1 v_2 : v_1 \neq v_2 \wedge \neg E(v_1, v_2)$

ii) $\forall v_1 v_2 : E(v_1, v_2)$

iii) $\forall v_1 v_2 : v_1 \neq v_2 \implies [E(v_1, v_2) \vee \exists v_3 : E(v_1, v_3) \wedge E(v_3, v_2)]$

iv) $\exists v_1 v_2 v_3 v_4 : v_2 \neq v_3 \wedge v_2 \neq v_4 \wedge v_3 \neq v_4 \wedge E(v_1, v_2) \wedge E(v_1, v_3) \wedge E(v_1, v_4)$

c) [6 pts] Write a logical statement like those in part (b) (involving only variables ranging over vertices, the expressions $E(v_i, v_j)$, $v_i = v_j$, $v_i \neq v_j$, logical connectives and quantifiers) which is satisfied precisely by those simple graphs which have a vertex of degree two.

Problem 3. [15 pts]

Q_n is the simple graph with 2^n vertices known as the n-dimensional hypercube. It can be defined recursively by the conditions that Q_0 consists of a single vertex and Q_{n+1} is obtained by taking two copies of Q_n and adding edges between each vertex in one copy and the corresponding vertex in the other copy.

a) [2 pts] How many edges are there in Q_3? ____

b) [8 pts] Describe a simple, closed-form formula for the number of edges in Q_n. ________

c) [5 pts] Prove below the correctness of your answer to part (b).

Problem 4. [15 pts]

Start with 100 coins on a table, 98 showing "heads" and two showing "tails." There are two ways to change the coins:

(i) flip over any ten coins, or

(ii) let n be the number of coins on the table showing heads. Place $n + 1$ additional coins on the table, with all the new coins initially showing tails.

For example, you might begin by flipping eight heads and two tails, yielding 92 heads and 8 tails, then add 93 tails, yielding 92 heads and 101 tails.

a) [3 pts] Model this situation as a state machine, carefully defining the set of states, the start state and the possible state transitions.

b) [2 pts] Explain how to reach a state with exactly one tail showing.

c) [4 pts] Define the following derived variables of a state q:

i) $C(q) =$ the number of coins on the table
ii) $H(q) =$ the number of heads
iii) $T(q) =$ the number of tails
iv) $C_2(q) = C(q) \bmod 2$
v) $H_2(q) = H(q) \bmod 2$
vi) $T_2(q) = T(q) \bmod 2$

Which of these variables is

i) increasing ____________________

ii) nondecreasing ____________________

iii) decreasing ____________________

iv) constant ____________________

d) [6 pts] Prove that it is not possible to reach a state in which there is exactly one head showing.

Problem 5. [10 pts]

Derive a closed form formula for

$$\sum_{i=0}^{n} \sum_{j=0}^{i} x^j$$

Problem 6. [15 pts]

a) [7 pts] Prove that $\log(n!) = \Theta(n \log n)$.

b) [8 pts] Prove or disprove: For functions f, g from positive integers to positive integers,

$$f(n) = O(g(n)) \implies 2^{f(n)} = O(2^{g(n)}).$$

Appendix

K_n is the "complete" graph with n vertices. It has an edge between every two vertices.

$K_{n,m}$ is the "bipartite" graph with $n+m$ vertices. Each one of the first n vertices is adjacent to every one of the last m vertices.

C_n is the simple cycle of n vertices.

W_n is the "wheel" graph. It consists of a simple cycle of n vertices, along with an $n+1$st vertex adjacent to each vertex on the cycle.

Q_n is the n-dimensional hypercube (see Problem 3). It has 2^n vertices.

L_n is the simple path of length n (see Problem 1). It has n edges and $n+1$ vertices.