Massachusetts Institute of Technology | Tutorial 9
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Problem Solutions for Tutorial 9

Problem 1. *Find a $\Theta(\cdot)$ bound for*

$$T(n) = T(\sqrt{n}) + \lg\lg n.$$

The base condition is $T(2) = 1$.

$$T(n) = T(\sqrt{n}) + \lg\lg n$$

Substitute $n = 2^m$ and note that $m = \lg n$.

$$T(2^m) = T(2^{\frac{m}{2}}) + \lg m$$

Let $S(m) = T(2^m)$, noting that the base case is $S(m=1) = T(2)$ so

$$\begin{aligned}
T(n) = T(2^m) = S(m) &= S(\frac{m}{2}) + \lg m \\
&= S(\frac{m}{2}) + \lg m \\
&= \lg m + \lg(m/2) + \lg(m/4) \cdots \lg(2) + S(1) \\
&= \sum_{i=0}^{(\lg m)-1} (\lg \frac{m}{2^i}) + S(1) \\
&= \sum_{i=0}^{(\lg m)-1} (\lg m - i) + S(1) \\
&= (\lg m - 1)(\lg m) - \sum_{i=0}^{(\lg m)-1} i + 1 \\
&= (\lg m - 1)(\lg m) - \frac{(\lg m - 1)(\lg m)}{2} + 1 \\
&= \frac{(\lg m - 1)(\lg m)}{2} + 1 \\
&= \frac{(\lg\lg n - 1)(\lg\lg n)}{2} + 1
\end{aligned}$$

So, $T(n)$ is $\Theta((lg \lg n)^2)$.

Problem 2. *Consider the linear recurrence*
$T(n) = \sum_{i=1}^{d} a_i T(n/2^i) + G(n)$

- *Find a transformation $n \rightarrow m$ that will convert this problem io a difference equation of the form*
  $f(m) = \sum_{i=1}^{d} a_i f(m-i) + g(m)$

  Let $n = 2^m$ and $f(m) = T(2^m)$.
- *Apply your results to find the solution of the recurrence relation*
  $T(n) = 2T(n/2) + 3T(n/4) + nlogn$
  Find the homogeneous solution first:
  Guess a solution of $f(m) = cx^m$. $cx^m = 2cx^{m-1} + 3cx^{m-2}$. $x^2 - 2x^1 - 3x = 0$. And the solutions are $x = -3, 1$.
  So the solution is $T(2^m) = f(m) = C_1(-3)^m + C_2$.
  And the overall solution is $T(n) = T(2^m) = C_1(-3)^{\lg n} + C_2 + Dn \log n$.

Problem 3. *Compute the following limits.*

- $\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$.
  Using partial fractions, we find that $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$.

$$\sum_{i=1}^{\infty}(\frac{1}{i} - \frac{1}{i+1}) = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) + \ldots$$

$$= 1 - \lim_{n \rightarrow \infty} \frac{1}{n} = 1$$

  This series is known as a telescoping summation since pairs of terms cancel.
- $\lim_{n \rightarrow \infty} 1/n + 1/(n+1) + \ldots + 1/2n$
  $1/n + 1/(n+1) + \ldots + 1/2n) \leq \int_{n-1}^{2n} \frac{1}{n} dn = \ln n|_{n-1}^{2n} = \ln(\frac{2n}{n-1})$
  $1/n + 1/(n+1) + \ldots + 1/2n) \geq \int_{n}^{2n+1} \frac{1}{n} dn = \ln n|_{n}^{2n+1} = \ln(\frac{2n+1}{n})$
  $\lim_{n \rightarrow \infty} \ln(\frac{2n+1}{n}) = 2 \lim_{n \rightarrow \infty} \ln(\frac{2n}{n-1}) = 2$
  So the sum has a tight bound of 2.