Binomial Coefficients

Review sessions 5-7 and 7-9 in 3-270 on 4/22.

Exam in 3-270 and 3-370 as before.

Go to 3-270 if you were born on an odd day of the month, 3-370 otherwise.

Should some office hours move to tuesday next week?

Readings: Section 4-3, 5-4
Next: 4-6, 4-7

Last time, we introduced the *binomial coefficients*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

These represent the number of ways to choose r elements from a set of n elements, where the order does not matter.
If $r > n$, define $\binom{n}{r} = 0$. Makes sense in the combinatorial interpretation. Also makes algebra simpler (see Vandermonde's identity later)
Combinations, as opposed to *permutations*, where the order does matter.
(There are

$$\frac{n!}{(n-r)!}$$

of those.)

Today, we study basic properties of binomial coefficients,
And finally revisit the principle of inclusion and exclusion, using binomial coefficients to prove it. Then an important theorem called the *Binomial Theorem*,

make an identity board

1 Basic Properties of Binomial Coefficients

Let n and r be naturals, $0 \leq r \leq n$.
Then

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Special cases:

$$\binom{n}{1} = n$$

$$\binom{n}{0} = 1$$

(uses the convention that $0! = 1$), (coincides with permutation model—1 permutation of no items)

$$\binom{n}{n} = 1$$

(note symmetry with previous),

$$\binom{n}{n-1} = n$$

(symmetry).

There are infinitely many interesting identities involving binomial coefficients.
You can't learn them all, but you can learn the basic ones and look others up/derive them as needed.
Can prove them by algebra, or by "thought experiments" that establish a combinatorial meaning for the LHS and RHS and argue that they are the same.

These identities are important because of how often binomial coefficients arise in analysis of algorithms (6.046) and probability (next week).

Theorem 1.1 Suppose $0 \leq r \leq n$.
Then

$$\binom{n}{r} = \binom{n}{n-r}.$$

Proof. (Algebraic)

$$\frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-(n-r))!}.$$

∎

Proof. (Combinatorial thought experiment)
LHS is number of ways to choose r from n.
RHS is number of ways to choose $n-r$.
Choosing a set of r is equivalent to choosing a set of $n-r$
(the elements *not* in the first chosen set)
– so there are just as many ways to choose either. ∎

The next theorem gives the ratio between successive binomial coefficients (same n, successive r):

Theorem 1.2 Suppose $1 \leq r \leq n$.
Then
$$\binom{n}{r} \Big/ \binom{n}{r-1} = (n-r+1)/r.$$

Proof. (Algebraic)
LHS =
$$\frac{n!}{r!(n-r)!} \times \frac{(r-1)!(n-(r-1))!}{n!} = (n-r+1)/r.$$
∎

Note that this ratio is > 1, which means the successive coefficients are increasing, as long as $r < \frac{n+1}{2}$.
Also, the ratio is < 1, which means the coefficient are decreasing, when $r > \frac{n+1}{2}$.
This says that the coefficients increase, then decrease, i.e., they are *unimodal*.
Form a bell-shaped curve:

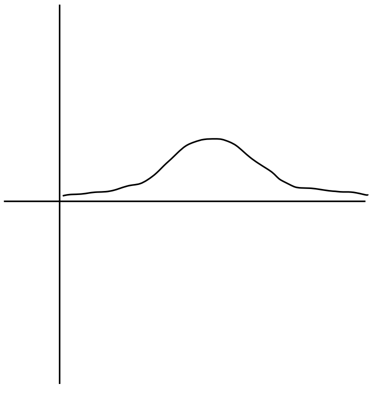

Figure 1: Bell Curve

In fact, the binomial coefficients form a real *bell curve*.
Like on tests:

Suppose that an SAT test has n questions.
Then the bell curve describes, for every r, the number of ways the student can get exactly r of the questions right, that is,

$$\binom{n}{r}.$$

If we suppose that each student taking the test is equally likely to get each of the n questions right or wrong, $\frac{1}{2}$ for each, then these coefficients turn out to be proportional to the number of students that get those numbers of questions right.
(See this when we do probability.)

Another identity:

Theorem 1.3 Suppose $1 \leq r \leq n$.
Then

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}.$$

Proof. (Algebraic **LTTR**)

$$\begin{aligned}
\binom{n}{r} &= \frac{n!}{r!(n-r)!} \\
&= \frac{n}{r}\frac{(n-1)!}{(r-1)!(n-r)!} \\
&= \frac{n}{r}\frac{(n-1)!}{(r-1)!((n-1)-(r-1))!} \\
&= \frac{n}{r}\binom{n-1}{r-1}
\end{aligned}$$

■

Proof. (Combinatorial)
Consider choosing a committee of size r and a leader, from n people.
One way is to pick the leader first (n ways), then pick the other $r-1$ committee members from the remainder ($\binom{n-1}{r-1}$ ways).
Another way is to pick the r committee members first ($\binom{n}{r}$ ways) and then pick a leader from among them (r ways).
This implies (by product rule) that:

$$n\binom{n-1}{r-1} = \binom{n}{r}r.$$

Now divide both sides by r. ■

Now an important theorem due to Pascal:

Theorem 1.4 (Pascal) Suppose $1 \leq r \leq n-1$.
Then

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

Probably most important identity, since gives a kind of recurrence for binomial coefficients.
Memorize!

Proof. (Algebraic, **LTTR**)

$$\begin{aligned}
\binom{n-1}{r} + \binom{n-1}{r-1} &= \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-r)!} \\
&= (n-r)\frac{(n-1)!}{r!(n-r)!} + r\frac{(n-1)!}{r!(n-r)!} \\
&= n\frac{(n-1)!}{r!(n-r)!} \\
&= \frac{n!}{r!(n-r)!} \\
&= \binom{n}{r}
\end{aligned}$$

∎

Proof. (Combinatorial—case analysis and sum rule) Let $S = \{1, \ldots, n\}$.
Let A be the set of r-element subsets of S.
Let B be the set of r-element subsets of S that contain 1.
Let C be the set of r-element subsets of S that don't contain 1.

Then $A = B \cup C$, and B and C are disjoint.
So $|A| = |B| + |C|$, by sum rule.
But now can get expressions for $|A|$, $|B|$ and $|C|$ as numbers of combinations:

- $|A| = \binom{n}{r}$.
- $|B| = \binom{n-1}{r-1}$.
  This is because $r-1$ other elements must be chosen from the remaining $n-1$.
- $|C| = \binom{n-1}{r}$.
  This is because r elements must be chosen from the remaining n.

So

$$\binom{n}{r} = |A| = |B| + |C| = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

■

Pascal's theorem has a nice pictorial representation:
Row represents n, starting with 0 in top row.
Successive elements represent r, starting with 0 at left.

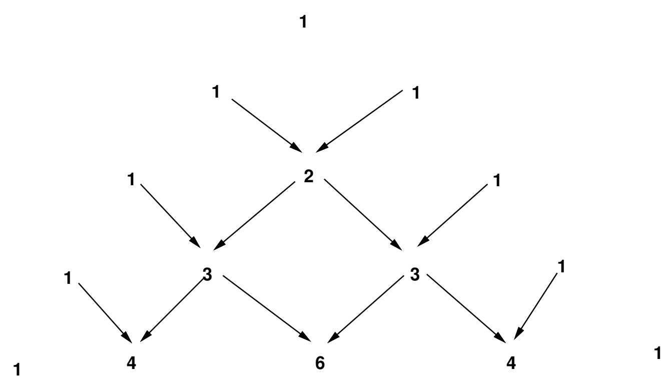

Figure 2: Pascal's triangle.

This triangle has lots of nice properties.
Experiment with it.
E.g., what happens if we add the coefficients in one row?

Theorem 1.5 Suppose n is any natural.
Then

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

Proof. (Algebraic). We show since demonstrates and induction and Pascal's identity.
Induction on n.
Let $P(n)$ be:

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

BASE: $n = 0$.

$$\sum_{r=0}^{0} \binom{0}{r} = \binom{0}{0} = 1 = 2^0$$

INDUCTIVE STEP: Assume $P(n)$, show $P(n+1)$, where $n \geq 0$.

$$\sum_{r=0}^{n+1} \binom{n+1}{r} = \tbinom{n+1}{0} + \textstyle\sum_{r=1}^{n} \tbinom{n+1}{r} + \tbinom{n+1}{n+1}$$

$$= 2 + \sum_{r=1}^{n} \binom{n+1}{r}$$

$$
\begin{aligned}
&= 2 + \sum_{r=1}^{n} \left(\binom{n}{r} + \binom{n}{r-1} \right) && \text{by Pascal's identity} \\
&= 2 + 2\sum_{r=1}^{n-1} \binom{n}{r} + \binom{n}{0} + \binom{n}{n} \\
&= 2\sum_{r=0}^{n} \binom{n}{r} \\
&= 2 \cdot 2^n && \text{by inductive hypothesis,} \\
&= 2^{n+1}.
\end{aligned}
$$

■

Proof. (Combinatorial)
There are 2^n different subsets of a set of n elements.
Decompose this collection based on the sizes of the subsets.
There are $\binom{n}{r}$ subsets of size r, for $r = 0, 1, \ldots, n$.
Hence the theorem follows. ■

Theorem 1.6 (Vandermonde) Suppose $0 \leq r \leq m, n$.
Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$$

Proof. (Combinatorial—case analysis, sum and product rules)
Suppose there are m red balls and n blue balls, all distinct.
There are $\binom{m+n}{r}$ ways to choose r balls from the two sets combined.
That's the LHS.
Now decompose this collection of choices based on how many of the chosen balls are blue.
For any k, $0 \leq k \leq r$, there are $\binom{m}{r-k}$ ways to choose $r-k$ red balls, and $\binom{n}{k}$ ways to choose k blue balls.
By the product rule, that makes $\binom{m}{r-k}\binom{n}{k}$ ways to choose r balls such that k of them are blue.
Adding up these numbers for all k gives the RHS. ■

LTTR: It is also possible to prove this algebraically, using Pascal's identity.
To do this, it's useful to generalize the notion of binomial coefficient a little bit:
If $r > n$, define $\binom{n}{r}$ to be 0.
Makes some kind of sense informally, because there aren't any ways to choose more elements than there are in a set!

We can then observe that some of the binomial identities are still correct with this extension.

E.g., Pascal's theorem is still true if $r > n$ (reduces trivially).
So we will prove Vandermonde's theorem without the restriction $r \leq m, n$.
This means that some of the terms in the statement of the theorem could be trivial terms, e.g., $\binom{n}{k}$ could occur, where $k > n$ (value taken as 0).

Proof. (Algebraic)
Induction using Pascal's identity:
Let $P(n)$ be

$$\forall m, r, \binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

BASE: $n = 0$. Then

$$\binom{m+0}{r} = \binom{m}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{0}{k}.$$

This latter equation is true because the only value of k for which the last factor isn't 0 is $k = 0$.
Hence $P(0)$ is true.

INDUCTIVE STEP: Assume $P(n)$ is true, show $P(n+1)$. We have:

$$\begin{aligned}
\binom{m+(n+1)}{r} &= \binom{m+n}{r} + \binom{m+n}{r-1} && \text{by Pascal's identity (don't have to worry about bounds)} \\
&= \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k} + \sum_{k=0}^{r-1} \binom{m}{r-1-k}\binom{n}{k} && \text{by inductive hypothesis (don't have to worry about bounds)} \\
&= \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k} + \sum_{k=1}^{r} \binom{m}{r-k}\binom{n}{k-1} && \text{by change of vars} \\
&= \binom{m}{r}\binom{n}{0} + \sum_{k=1}^{r} \binom{m}{r-k}(\binom{n}{k} + \binom{n}{k-1}) && \\
&= \binom{m}{r}\binom{n}{0} + \sum_{k=1}^{r} \binom{m}{r-k}\binom{n+1}{k} && \text{by Pascal's identity} \\
&= \binom{m}{r}\binom{n+1}{0} + \sum_{k=1}^{r} \binom{m}{r-k}\binom{n+1}{k} && \\
&= \sum_{k=0}^{r} \binom{m}{r-k}\binom{n+1}{k}
\end{aligned}$$

Hence, $P(n+1)$ is true. So $P(n)$ is true for all n. ∎

2 The Binomial Theorem

An important theorem giving the coefficients in the expansion of powers of binomial expressions.
Turns out they are binomial coefficients.

2.1 The Theorem

Theorem 2.1 Suppose $n \in \mathbb{N}$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Example 1 $(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

Proof. (Algebraic)
By induction, Pascal's identity (HW). ∎

Proof. (Combinatorial)

$$(x+y)(x+y)(x+y)(x+y)\dots$$

n factors.
Each $x^k y^{n-k}$ term selects k binomials from the n to have x, rest y.
$\binom{n}{k}$ ways of choosing the binomials to have x. ∎

2.2 Identities via Binomial Theorem

The binomial theorem can be used to give slick proofs of some binomial identities, e.g.:

Theorem 2.2

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof. Plug $x = 1$, $y = 1$ into binomial theorem. ∎

Theorem 2.3

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Proof. Plug $x = -1$, $y = 1$ into binomial theorem. ■

Combinatorial interpretation: number of ways of selecting an even number of elements from a set of n equals the number of ways of selecting an odd number.
Works for even and odd!

Example 2 $n = 5$:
Choosing odd is $\binom{5}{1} + \binom{5}{3} + \binom{5}{5} = 5 + 10 + 1 = 16$.
Choosing even is $\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 1 + 10 + 5 = 16$.

Example 3 $n = 6$:
Choosing odd is $6 + 20 + 6 = 32$.
Even is $1 + 15 + 15 + 1 = 32$.

For odd set, intuitive (choosing even set leaves odd set). For even set, may be somewhat surprising—no obvious bijection between even and odd sets.

Theorem 2.4

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} = 2^{n-1}.$$

Proof. (Algebraic **LTTR**)
Add the equations in the two previous theorems.
Get

$$\sum_{k=0}^{n} (\binom{n}{k} + (-1)^k \binom{n}{k}) = 2^n.$$

Odd terms cancel out, even terms double, so get:

$$2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} = 2^n.$$

Then divide by 2. ■

Proof. (Combinatorial)
LHS represents number of ways to pick an even number of elements from a set of n elements.
The number of ways to pick an even number is equal to the number of ways to pick an odd number of elements.
Since the total number of ways is 2^n, the number of ways to pick even is half this, or 2^{n-1}. ■

A new proof of Pascal's identity:

Proof. $(x+y)^n = x(x+y)^{n-1} + y(x+y)^{n-1}$.
Consider coefficient of $x^r y^{n-r}$.
LHS: $\binom{n}{r}$.
RHS: Obtained by adding coefficient of $x^{r-1}y^{n-r}$ in the first term, plus the coefficient of $x^r y^{n-1-r}$ in the second term.
These are $\binom{n-1}{r-1}$ and $\binom{n-1}{r}$, respectively. ∎

Vandermonde:

Proof. $(1+t)^{m+n} = (1+t)^m(1+t)^n$.
Consider coefficient of t^r.
LHS: $\binom{m+n}{r}$.
RHS: Obtained by adding up, for all k, the product of the coefficient of t^k in the second factor and the coefficient of t^{r-k} in the first factor.
This is

$$\Sigma_{k=0}^{r}\binom{n}{k}\binom{m}{r-k}.$$

∎

2.3 General Binomial Theorem

Can generalize binomial coefficients once again, allowing the first argument to be any real number:

Definition 2.5 General binomial coefficient $\binom{\alpha}{k}$ for $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$:

$$\binom{\alpha}{k} = \tfrac{1}{k!}\prod_{j=0}^{k-1}(\alpha - j) = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

$$\binom{\alpha}{0} = 1$$

This doesn't have the same kinds of combinatorial meaning as before, but works out nice mathematically.
In particular, a version of the binomial theorem holds:

Theorem 2.6 Let $\alpha \in \mathbb{R}$ and $\left|\frac{x}{y}\right| < 1$. Then

$$(x+y)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k y^{\alpha-k}$$

Proof. Calculus. ∎

Example 4 Approximate $\sqrt{2}$ using binomial theorem.

$$\sqrt{2} = (x+y)^{\frac{1}{2}}, \text{ where } x = 1, y = 1$$

However, can't use b.t. directly, because

$$\left|\frac{x}{y}\right| \not< 1$$

.
However,

$$\sqrt{2} = 2\frac{\sqrt{2}}{2} = 2\sqrt{\frac{1}{2}} = 2(1-\frac{1}{2})^{\frac{1}{2}} = 2(x+y)^{\frac{1}{2}}, \text{ where } x = 1, y = -\frac{1}{2}$$

$$(1-\frac{1}{2})^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \left(-\frac{1}{2}\right)^k$$

$$\binom{\frac{1}{2}}{0} = 1, \binom{\frac{1}{2}}{1} = \frac{1}{2}, \binom{\frac{1}{2}}{2} = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!} = -\frac{1}{8}, \binom{\frac{1}{2}}{3} = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} = \frac{1}{16}, etc.$$

$$\begin{aligned}\sqrt{2} &= 2\left(1 + \tfrac{1}{2}\left(-\tfrac{1}{2}\right) + \left(-\tfrac{1}{8}\right)\left(\tfrac{1}{4}\right) + \left(\tfrac{1}{16}\right)\left(-\tfrac{1}{8}\right) + \cdots\right) \\ &= 2\left(1 - \tfrac{1}{4} - \tfrac{1}{32} - \tfrac{1}{128} - \cdots\right) \\ &\approx 2(.7109375) = 1.421875\end{aligned}$$

3 Principle of Inclusion and Exclusion

Finally, we return to the principle of inclusion-exclusion from last lecture.
Use binomial coefficients to help prove this, via new alternating sum identity proved by binomial theorem.
Recall (for general m):

Theorem 3.1 $|\cup_i A_i| = \Sigma_i |A_i| - \Sigma_{i,j \text{ distinct}} |A_i \cup A_j| + \Sigma_{i,j,k \text{ distinct}} |A_i \cup A_j \cup A_k| - \ldots + (-1)^{m+1} |\cap A_i|$

Can be proved by induction on the number of sets – LTTR.
Now we will give a combinatorial proof using binomial coefficients.

Proof. (Combinatorial)
Each term in the summation counts certain elements in $\cup_i A_i$.
We prove that every element of this union is counted exactly once.
So, consider any particular element a, and suppose it is in exactly r of the sets A_i.
We see how many times a is counted by each term.

In the first term, a is counted $r = \binom{r}{1}$ times, once for each of the sets that contains it.
In the second term, a is counted $\binom{r}{2}$ times, once for each of the pairs of sets that contain it.
In the third term, a is counted $\binom{r}{3}$ times, ...,
and in the rth term, a is counted $\binom{r}{r} = 1$ times.
The terms alternate signs, so the total number of times a is counted is:

$$\binom{r}{1} - \binom{r}{2} + \binom{r}{3} - \ldots - (-1)^r \binom{r}{r}.$$

But Theorem 2.3 implies that

$$\sum_{k=0}^{r} (-1)^k \binom{n}{k} = 0.$$

This implies that the sum above is equal to $\binom{r}{0} = 1$.
That is, a is counted exactly once, as needed. ∎