Problem Set 13 Solutions

Problem 1.

(a) Observe that there are $6^3 = 216$ possibilies for the values on 3 dice.

1. The number of ways to get 3 different values is $6 \cdot 5 \cdot 4 = 120$. So the probability is $\frac{120}{216} = \frac{5}{9}$.
2. The number of ways to get two identical values is $\binom{3}{2}(6 \cdot 1) \cdot 5 = 90$. So the probability is $\frac{90}{216} = \frac{5}{12}$.
3. The number of ways to get one value is 6. So the probability is $\frac{6}{216} = \frac{1}{36}$.

(b) If all three dice have different values, the net winning is \$3 - \$3 = \$0. If two dice have the same value (and the third has a different one), the net winning is \$2 + \$1 - \$4 = -\$1. If all dice have the same value, the net winning is \$3 - \$5 = -\$2. So the expected return is $\frac{5}{9} \cdot 0 + \frac{5}{12} \cdot (-1) + \frac{1}{36}(-2) = -\$\frac{17}{36}$.

(c) Let X_i be the random variable whose value is the net winning if we wager \$1 on the value i. Note that $X_1 + X_2 + X_3 + X_4 + X_5 + X_6$ is the random variable whose value is the net winning if we wager \$1 on each of the 6 values. Now $\mathbf{E}[X_1] = \mathbf{E}[X_2] = \cdots = \mathbf{E}[X_6]$, so by the linearity of expectation, the expected winning is $\frac{1}{6} \cdot (-\frac{17}{36}) = -\$\frac{17}{216}$.

(d) The game will be fair if $\mathbf{E}[X_i] = 0$. This is the same as requiring $\mathbf{E}[X_1 + \cdots + X_6] = 0$, i. e., that the game described in the second part of the question is fair. If all three dice have different values, the net winning is $3i_1 - 3$. If two dice have the same value (and the third has a different one), the net winning is $2i_2 + i_3 - 4$. If all dice have the same value, the net winning is $i_3 - 5$. So the expected return is $\frac{5}{9} \cdot (3i_1 - 3) + \frac{5}{12} \cdot (i_1 + i_2 - 4) + \frac{1}{36}(i_3 - 5) = \frac{25}{12}i_1 + \frac{5}{12}i_2 + \frac{1}{36}i_3 - \frac{125}{36}$. For the game to be fair the expectation has to be zero. So this translates as $75i_1 + 15i_2 + i_3 = 125$. A possible choice would be $(i_1, i_2, i_3) = (1, 3, 5)$. (There are other solutions.)

Problem 2.

(a) At any step, the probability that there is a failure in either component is $p + p - p^2 = 2p - p^2$. So $\mathbf{E}[X] = 1/(2p - p^2)$.

(b) At any step, the probability that there is a failure in both components is p^2. So $\mathbf{E}[Y] = 1/p^2$.

Problem 3.

(a) Let X_i be the random variable whose value is 1 if student i is ranked higher on list from grad school B than for grad school A and 0 otherwise. Now if student i has rank r for grad school B, then the probability that his/her rank for grad school A is lower is $(r-1)/n$. In other words, we know that **Pr**[student i has higher rank on the list for grad school B| student i has rank r on grad school B] $= (r-1)/n$. Now **Pr**[student i has rank r on grad school B] $= 1/n$. So by the theorem of total probability, **Pr**[student i has higher rank on list for grad school B] $= \sum_{r=1}^{n} \frac{r-1}{n} \cdot \frac{1}{n} = \frac{n-1}{2n}$.

By the linearity of expectation, the expected number of students that have a higher rank on the list for grad school B is $\mathbf{E}[X_1] + \cdots + \mathbf{E}[X_n] = \frac{n-1}{2}$.

Another way to do this is to observe that the expected number of students with higher rank is the same as the number with lower rank. The expected number of students with the same rank, by linearity of expectation, is 1. (The probability that student i is ranked the same in both schools is $1/n$.) This gives us the same answer as before.

(b) Let X_i be the random variable whose value is 1 if student i is ranked at least k higher on list from grad school B than for grad school A and 0 otherwise. Now if student i has rank r (where $r > k$) for grad school B, then the probability that his/her rank for grad school A is at least k lower is $(r-k)/n$. In other words, we know that **Pr**[student i has at least k higher rank for grad school B | student i has rank r on grad school B] $= (r-k)/n$. Now **Pr**[student i has rank r on grad school B] $= 1/n$. So by the theorem of total probability, **Pr**[student i has higher rank on list for grad school B] $= \sum_{r=k}^{n} \frac{r-k}{n} \cdot \frac{1}{n} = \frac{(n-k)(n-k+1)}{2n^2}$.

By the linearity of expectation, the expected number of students that have a higher rank on the list for grad school B is $\mathbf{E}[X_1] + \cdots + \mathbf{E}[X_n] = \frac{(n-k)(n-k+1)}{2n}$.

Problem 4.

(a) Let X_i be the indicator random variable whose value is 1 if box i is empty and is 0 otherwise. Now the probability that box i is empty is the probabilility that all n balls land in all other boxes, which is $(1-\frac{1}{n})^n$. So $\mathbf{E}[X_i] = (1-\frac{1}{n})^n$. By the linearity of expectation, the expected number of empty boxes is $\mathbf{E}[X_1] + \cdots + \mathbf{E}[X_n] = n(1-\frac{1}{n})^n$.

(b) For any subset S of k balls, let $\mathbf{E}[S]$ be the event that each of these balls falls in box 1. There are $\binom{n}{k}$ such events, corresponding to each subset of k balls, and these events intersect. We are intestested in the probability of the union of these events, which is at most the sum of their individual probabilities: $\mathbf{Pr}[\cup_S \mathbf{E}[S]] \leq \sum_S \mathbf{Pr}[\mathbf{E}[S]]$. For any fixed set S, $\mathbf{Pr}[\mathbf{E}[S]] = \left(\frac{1}{n}\right)^k$, since we don't care where other balls land. Thus, the probability that at least k balls fall in box 1 is at most $\binom{n}{k}\left(\frac{1}{n}\right)^k$.

Expanding $\binom{n}{k}$, we get $\binom{n}{k}\frac{1}{n^k} = \frac{n(n-1)\cdots(n-k+1)}{k!\,n^k} = \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n} \cdot \frac{1}{k!} \leq \frac{1}{k!}$ Summing over the different boxes, we have

$\mathbf{Pr}[\geq k$ balls fall into *some* box$] \leq n/k!$.

(c) $R \geq k$ exactly when some box has at least k balls, so using Stirling's formula,

$$\mathbf{Pr}[R \geq k] \leq n \frac{1}{\sqrt{2\pi k}} \left(\frac{e}{k}\right)^k .$$

(d) When $k = n^\alpha$ (for some constant α),

$$\mathbf{Pr}[R \geq k] \leq n \cdot \frac{1}{\sqrt{2\pi n^\alpha}} \cdot \frac{e^{n^\alpha}}{(n^\alpha)^{n^\alpha}} = \frac{1}{\sqrt{2\pi}} \cdot e^{-n^\alpha(\alpha \ln n - 1) + (1-\alpha/2)\ln n} = e^{-\Theta(n^\alpha \ln n)}$$

which is exponentially small in n.

When $k = \lg\lg n$,

$$\begin{aligned}
\mathbf{Pr}[R \geq k] &\leq n \cdot \frac{1}{\sqrt{2\pi \lg\lg n}} \cdot \frac{e^{\lg\lg n}}{(\lg\lg n)^{\lg\lg n}} \\
&= \frac{1}{\sqrt{2\pi}} \cdot e^{\ln n - \frac{1}{2}\ln\lg\lg n} \cdot \frac{e^{\lg\lg n}}{e^{(\ln\lg\lg n)(\lg\lg n)}} \\
&= e^{\Theta(\ln n - \frac{1}{2}\ln\lg\lg n + \lg\lg n - \ln\lg\lg n \lg\lg n)} \\
&= e^{\Theta(\ln n)} ,
\end{aligned}$$

which is much greater than 1, trivializing our upper bound.

(e) We would like to know what value of $k = k(n)$ (that is, k as a function of n) makes $n \cdot \frac{1}{\sqrt{2\pi k}} \left(\frac{e}{k}\right)^k$ equal to a constant. Asymptotically, we would like $e^{\ln n} \frac{e^k}{k^k}$ to be some constant. It is clear that for $k = \omega(\lg n)$, the denominator is growing faster than the numerator and this expression is asmyptotically 0. When $k = o(\lg n)$, the numerator's growth is governed by $e^{\ln n} = n$, so we really want to find out when n/k^k is a constant.

Solving $n/k^k = 1$ (or any constant c instead of 1) by taking logs of both sides, we get $k \lg k = \lg n$. Iterating this once, we get
$k = \lg n / \lg k = \lg n / \lg(\lg n / \lg k) = \lg n / (\lg\lg n - \lg\lg k) \approx \lg n / \lg\lg n$ since $k = o(\lg n)$.

It is easy to check that $k \approx \frac{\lg n}{\lg\lg n}$ satisfies $k^k = n$:

$$\begin{aligned}
\left(\frac{\lg n}{\lg\lg n}\right)^{\frac{\lg n}{\lg\lg n}} &= 2^{\left(\lg\left(\frac{\lg n}{\lg\lg n}\right)\right)\frac{\lg n}{\lg\lg n}} \\
&= 2^{(\lg\lg n - \lg\lg\lg n)\frac{\lg n}{\lg\lg n}} \\
&= 2^{\lg n - \lg n\left(\frac{\lg\lg\lg n}{\lg\lg n}\right)} \\
&= n \cdot n^{o(1)}.
\end{aligned}$$

Problem 5.

Let X be the random variable whose value is the roll in the first stage. Let $Y_1, Y_2, \ldots, Y_X$ be the random variable whose values are the rolls in the second stage. Define $Y = Y_1 + \cdots + Y_X$.

Let $Z_1, Z_2, \ldots, Z_Y$ be the random variable whose values are the rolls in the third stage. Define $Z = Z_1 + \cdots + Z_Y$. We want to find $\mathbf{E}[Z]$. Now all the rolls are mutually independent, so we can use Wald's theorem to get $\mathbf{E}[Y] = \mathbf{E}[X]\mathbf{E}[Y_i] = (\frac{7}{2})^2$. Since Y is determined by rolls that are independent of the rolls in the third stage, Y and Z_i are independent. So we apply Wald's theorem again to get $\mathbf{E}[Z] = \mathbf{E}[Y]\mathbf{E}[Z_i] = (\frac{7}{2})^3$.

The answer is $(\frac{7}{2})^3$.

Problem 6.

(a) Observe that $\mathbf{Pr}[\max(X,Y) = i] = \mathbf{Pr}[X = i \text{ and } Y < i] + \mathbf{Pr}[X < i \text{ and } Y = i] + \mathbf{Pr}[X = Y = i] = \frac{1}{n} \cdot \frac{i-1}{n} + \frac{i-1}{n} \cdot \frac{1}{n} + \frac{1}{n} \cdot \frac{1}{n} = \frac{2i-1}{n^2}$.
This means that $\mathbf{E}[\max(X,Y)] = \sum_{i=1}^{n} \frac{i(2i-1)}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n}(2i^2 - i) = \frac{1}{n^2}(\frac{2n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}) = \frac{(n+1)(4n-1)}{6n}$.

(b) Observe that $\mathbf{E}[\max(X,Y)] + \mathbf{E}[\min(X,Y)] = \mathbf{E}[\max(X,Y) + \min(X,Y)] = \mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y]$. Now $\mathbf{E}[X] = \mathbf{E}[Y] = \frac{n+1}{2}$. so $\mathbf{E}[\min(X,Y)] = n + 1 - \frac{(n+1)(4n-1)}{6n} = \frac{(n+1)(2n+1)}{6n}$.

(c) Note that $\mathbf{Var}[X-Y] = \mathbf{Var}[X] + \mathbf{Var}[Y]$ and that $\mathbf{Var}[X] = \mathbf{Var}[Y] = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \sum_{i=1}^{n} \frac{i^2}{n} - (\frac{n+1}{2})^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}$. So $\mathbf{Var}[X-Y] = \frac{n^2-1}{6}$.

(d) Now $\mathbf{Var}[|X-Y|] = \mathbf{E}[|X-Y|^2] - (\mathbf{E}[|X-Y|])^2$. But $\mathbf{E}[|X-Y|^2] = \mathbf{E}[(X-Y)^2] = \mathbf{Var}[X-Y] + (\mathbf{E}[X-Y])^2 = \mathbf{Var}[X-Y] = \frac{n^2-1}{6}$ (since $\mathbf{E}[X-Y] = \mathbf{E}[X] - \mathbf{E}[Y] = 0$). So all we need is to compute $\mathbf{E}[|X-Y|]$.
But $|X-Y| = \max(X,Y) - \min(X,Y)$, so by linearity of expectation, $\mathbf{E}[|X-Y| = \mathbf{E}[\max(x,y)] - \mathbf{E}[\min(x,y)] = \frac{(n^2-1)}{3n}$.
This means that $\mathbf{Var}[|X-Y|] = \frac{n^2-1}{6} - (\frac{n^2-1}{3n})^2 = \frac{(n^2-1)(n^2+2)}{18n^2}$.

Problem 7.

(a) There are many examples to see that X and Y are not independent. An easy one is to note that $\mathbf{Pr}[X = 2]\mathbf{Pr}[Y = 1] \neq 0$ but $\mathbf{Pr}[XY = 2] = \mathbf{Pr}[X^3 = 2] = 0$.
To see that $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$, note that $\mathbf{E}[X] = 0$ and $\mathbf{E}[XY] = \mathbf{E}[X^3] = 0$ since X and X^3 are both symmetrical.

(b) We have that $\mathbf{Var}[X+Y] = \mathbf{E}[(X+Y)^2] - (\mathbf{E}[X+Y])^2 = \mathbf{E}[X^2 + 2XY + Y^2] - (\mathbf{E}[X] + \mathbf{E}[Y])^2 = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 + \mathbf{E}[Y^2] - (\mathbf{E}[Y])^2 + 2\mathbf{E}[XY] - 2\mathbf{E}[X]\mathbf{E}[Y] = \mathbf{Var}[X] + \mathbf{Var}[Y]$.

Problem 8.

(a) The expected number of heads of 50. So the probability that the number of heads is at least 70 is at most $50/70 = 0.71$.

(b) Let X_i be the random variable whose value is 1 if the ith coin flip is heads. Then $\mathbf{Var}[X_i] = 1/2 - (1/2)^2 = 1/4$. So $\mathbf{Var}[X_1 + \cdots + X_100] = 100/4 = 25$. The variance of the number of heads is $100/4 = 25$, so the standard deviation is 5. So 70 is four times the standard deviation from the mean. Since the distribution is symmetric, the probability is at most $\frac{1}{2} \cdot \frac{1}{4^2} = \frac{1}{32}$

(c) We apply Chernoff's bound with $c = 70/50 = 1.4$. This gives us that $\alpha = \ln(1.4) + 1/1.4 - 1 = 0.05076$ and that the probability is at most $e^{-0.05076 \cdot 1.4 \cdot 50} = 0.0286$.