Practice Final Exam

Problem 1. Let A be a set and R be a binary relation on A. For each of the statements in the box on the left below, indicate the number of the formula in the box on the right expressing that property. (Recall that the *composite* of two binary relations R and S, written $S \circ R$, is the relation $\{(a,c) \mid \text{there exists some } b \text{ such that } (a,b) \in R \text{ and } (b,c) \in S\}$. R^T is the *transpose* of R, i.e., $R^T = \{(a,b) \mid (b,a) \in R\}$, and I denotes the *identity* relation on A, i.e., $I = \{(a,a) \mid a \in A\}$.)

a.	R is transitive	
b.	R is antisymmetric	
c.	R is reflexive	
d.	R is symmetric	
e.	R is a bijection	
f.	R is an equivalence relation	
g.	R is a partial order	

1.	$R = R^T$
2.	$R \circ R \subseteq R$
3.	$R \cap R^T \subseteq I$
4.	$R \cap R^T \subseteq I \subseteq R \circ R \subseteq R$
5.	$I \subseteq R$
6.	$R \circ R^T = R^T \circ R = I$
7.	$I \subseteq R \circ R \subseteq R \subseteq R^T \cap R$

Problem 2. Recurrences.

(a) A vampire arrives in Boston at day 0 and starts biting people at day 1. People bitten become vampires themselves. New vampires bite two persons on the next day after they were bitten and four persons every day afterwards. Write an expression for the number of vampires in n days for $n \geq 2$, assuming the first vampire is a newly made vampire. *Do not solve the recurrence relation.*

(b) Consider the recurrence $T(n) = T(n/2) + T(n/4) + n$ and $T(1) = 1$. Prove that $T(n) \leq 2n$.

(c) Prove that $T(n) = \Theta(n)$ in the recurrence above.

Problem 3. **(a)** Six people communicate via radio frequency with one another. Each pair of people choose one of **two** available frequencies with which to communicate. Show that there is a subset of 3 people who are each using the identical frequency with which to communicate with others in the subset.

(b) Now suppose there are seventeen people communicating via radio frequency with one another. Each pair of people choose one of **three** available frequencies with which to communicate. Show that there is a subset of 3 people who are each using the identical frequency with which to communicate with others in the subset.

Problem 4. In this problem, A, B and C denote arbitrary sets. f and g are functions as specified in each part and I denotes the identity function. For each statement indicate whether it is **T**rue or **F**alse. No explanation is required.

T F There exists an injection (one-to-one) $f : A \to B$ if and only if there exists a surjection (onto) $g : B \to A$.

T F Let $f : A \to B$ and $g : B \to A$. $f \circ g = I$ implies $g \circ f = I$.

T F Let $f : A \to B$ and $g : B \to C$. If both f and g are injections, then $g \circ f$ is an injection.

T F Let $f : A \to B$ and $g : B \to C$. $g \circ f$ is a surjection implies both f and g are surjections.

T F Let $f : A \to B$ be an injection. If $|B| \leq |A|$, then f is a bijection.

Problem 5. Answer **T** or **F**:

T F The mapping $f_a : Z_p \to Z_p$ defined by

$$f_a(x) = ax \pmod p$$

is a bijection for any nonzero $a \in Z_p$, when p is prime.

T F For all a, b in $\mathbb{N}$, $\gcd(a^2b, ab^2) = ab$.

T F If $a^{-1} \pmod b$ exists, then so does $b^{-1} \pmod a$.

T F Let $\lambda(n)$ denote the number of distinct prime factors of n. Then $\lambda(n) = \Omega(\lg \lg n)$.

Problem 6. Answer **T** or **F**:

T F Adding k edges to an undirected graph increases its chromatic number (the number of colors needed to color the vertices) by at most k.

T F The number of length-3 paths in an n-vertex undirected graph is $O(n^2)$.

T F The number of length-3 paths in an n vertex tree is $O(n)$

T F Reversing the directions of all edges in a directed graph leaves the number of simple directed cycles unchanged.

T F If an undirected graph has the degree of every vertex divisible by a prime p, then p must evenly divide the number of edges.

Problem 7. Find the number of permutations of the letters $\{a, b, c, d, e, f, g\}$ that do not contain "bag" or "def".

Problem 8. Consider the following recursive algorithm for coloring a graph G with natural numbers. For a vertex v, $G - v$ denotes the graph you get by removing v and all its incident edges from G.

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Color(G)
    if G has 1 vertex v
        set the color of v to be 0
    else
        choose a vertex v of G
        Color(G − v)
        set the color of v to be the minimum n ∈ ℕ for which no neighbor of G has color n
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Prove that this algorithm legally colors G with a number of colors no more than 1 plus the maximum degree of a vertex in G.

Problem 9. For a certain test for a newly discovered virus, called JABOLA, it is known that 99% of the people carrying the virus will test positive, whereas only 1% of the people not carrying the virus will test positive. Suppose 0.5% of the population carries the JABOLA virus. Given that a person picked at random tests positive, what is the conditional probability that this person carries the JABOLA virus?

Problem 10. A lake contains n trout. 100 of them are caught, tagged and returned to the lake. Later another 100 trout are caught, selected independently from the first 100.

(a) Assume that the second 100 trout are selected without replacement. Write an expression (in n) for the exact probability that these trout are found to contain exactly 7 tagged ones.

(b) Now consider selecting the second 100 trout with replacement. That is, you repeat 100 times the following steps: select at random a trout in the lake, check if the trout is tagged and return it to the lake before selecting a new trout.

What is the probability that exactly 7 of the selected trout are tagged?

Problem 11. A bridge hand consists of 13 cards. The *suit-distribution* of a hand is the number of cards in each suit. For example, one possible suit-distribution is 4 spades, 3 hearts, 3 diamonds, and 3 clubs.

(a) How many different suit-distributions are there?

(b) How many different suit-distributions are there assuming that you get at least one card in each suit?

Problem 12. Suppose the entries a, b, c, d of a 2×2 matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

are chosen independently at random according to the uniform distribution on $\{0, 1, 2, 3, 4\}$. Let $X = \det(M) = ad - bc$.

(a) What is $E(X)$?

(b) What is $E(X \mid b = 1)$?

(c) What is $Var(X)$?

Problem 13. We consider an algorithm based on a given rooted, undirected tree, oriented with the root on the top. Every node in the tree has an associated *value* variable. Initially, the *value* variables of the leaves contain natural number input values, and the *value* variables of the root and other internal nodes contain a special "undefined" value $\perp$. The algorithm is supposed to end up with the root's *value* variable containing the sum of all the *value*s of all the leaves. This is accomplished by propagating summarized *value* information up the tree structure.

This algorithm is described precisely by the following state machine. This machine assumes a fixed undirected tree $T = (V, E)$, rooted at node r. The state components are:

for each $v \in V$:
$value(v) \in \mathbb{N} \cup \{\perp\}$; initially $value(v) \in \mathbb{N}$ if v is a leaf, and $value(v) = \perp$ if v is an internal node.

The transitions are:

$summarize(v)$, v an internal node
Precondition:
$value(v) = \perp$
for every $w \in children(v)$
$value(w) \in \mathbb{N}$
Effect:
$value(v) := \sum_{w \in children(v)} value(w)$

The property we are interested in is that $value(r)$ for the root r, if it ever gets defined, is equal to the sum of the *value*s of all the leaves. Formally:

Inv1: If $value(r) \in \mathbb{N}$, where r is the root, then $value(r) = \sum_{l \in leaves} value(l)$.

State and prove a strengthened version of Inv1, which can be proved by induction.