Massachusetts Institute of Technology
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Problem Set 12 Solutions

Problem 1. The Three-Cornered Duel.

A, B, and C are to fight a three-cornered pistol duel. They all know that A's chance of hitting the target is 0.3, C's is 0.5 and B never misses. They are to shoot in the sequence A, B, C and fire at their choice of target until one person is left standing. Show that A is better off missing deliberately the first shot than attempting to hit either B or C.

First consider a scenario where only A and C are left. Let S be the event that A survives if C shoots first.

$$P(S) = 0.5(0.3 + 0.7 * P(S)) \Rightarrow P(S) = 0.23$$

Let's calculate A's chances of survival under the three possible strategies.

1. If A shoots B first, then with probability 0.3 he kills him and then we are in the above scenario. If A misses B, then B kills C and than A has 0.3 chance to hit B and survive. So A's probability of survival under this strategy is $.3 * P(S) + .7 * .3 = .28$.
2. If A shoots C and kills him then he will be killed by B, while if he misses, B will kill C and then A has 0.3 probability to get B So A's probability of survival is $0.7 * .3 = 0.21$.
3. If A misses deliberately then B will kill C and A has probability 0.3 to survive.

So A best strategy is to shoot in the air. Under either strategy, A's chances are slim. I wonder how he got himself into that.

Problem 2. *If you repeatedly flip a fair coin, what is the probability you see the sequence HTT before you see the sequence HHT? (Hint: the answer is* **not** *1/2).*

First note that with an infinite number of tosses, we will eventually get either an HHT or an HTT with 4 exceptions: all heads, all tails, alternating heads and tails (starting with a head, and again starting with a tail). Note that this is 4 out of an infinite number of possible sequences and is therefore neglible.

Let P_{HTT} be the probability that we see the HTT sequence before we see the HHT sequence.

The probability that we see the HHT sequence before the HTT sequnces is therefore $P_{HHT} = 1 - P_{HTT}$.

Find P_{HHT}:

First let's assume that we start out with a head, H_S. The probabilities of getting the following sequences are:

- $P(H_S HH) = 1/4$
- $P(H_S HT) = 1/4$
- $P(H_S TH) = 1/4$
- $P(H_S TT) = 1/4$

Let Q be the probability of getting HHT when you start with a head.

- $H_S HH$
  Note that we get HHT unless we have an infinite number of heads (the probability of this is neglible though).
- $H_S HT$
  We have HHT.
- $H_S TH$
  We can get either HHT or HTT. We're in the same position as when we started– we see a head.
- $H_S TT$
  We have HTT.

$$
\begin{aligned}
Q &= P(H_S HH) + P(H_S HT) + P(H_S TH) \cdot Q \\
&= \frac{1}{4} + \frac{1}{4} + \frac{Q}{4} \\
&= \frac{1}{2} + \frac{Q}{4} \\
\frac{3}{4}Q &= \frac{1}{2} \\
Q &= \frac{2}{3}
\end{aligned}
$$

Now, if we start with a series of n tails (T), then when we see the first head (H), we know the probability of seeing HHT before HTT is Q, so the probability of getting HHT before HTT if we start with n tails is $(1/2)^n(1/2)Q$ (n tails at $p = 1/2$ each, followed by one head at $p = 1/2$).

Now, the probability of getting a head for the first throw is $1/2$, and the probability of getting a tail is $1/2$.

$$
\begin{aligned}
P_{HHT} &= \frac{1}{2}Q + \sum_{i=1}^{\infty}(\frac{1}{2})^{n+1}Q \\
&= \frac{1}{2}Q + \frac{1}{2}Q \\
&= Q \\
&= \frac{2}{3}
\end{aligned}
$$

So, $P_{HTT} = 1 - P_{HHT} = \frac{1}{3}$.

Problem 3. The Prisoner's Dilemma.

Three Prisoners A,B, and C have applied for parole. The parole board has decided to release two of the three, and the prisoners know this but not which two. A warder friend of prisoner A knows who are to be released. A realizes that it would be unethical to ask if he is going to be released, but thinks of asking for the name of one prisoner other than himself *who is to be released. Before he asks his chances are 2/3, but he thinks that if he asks and the warder says "B will be released", his own chances will become 1/2 since either A or C will be the only remaining candidates. So A decides not to reduce his chances asking. Is A's argument correct? Show what his chances are after asking.*

Let E_{AB}, E_{AC}, E_{BC} be the events that the corresponding prisoners are released. Each of these events had probability 1/3. Let $A = E_{AB} \cup E_{AC}$ be the event that A is released. We wish to evaluate $P(A|W_B)$ where W_B is the event that the warder says B

$$
\begin{aligned}
P(A|W_B) = \frac{P((E_{AB} \cup E_{AC}) \cap W_B)}{P(W_B)} &= \frac{P((E_{AB} \cap W_B) \cup (E_{AC} \cap W_B))}{P(W_B)} \\
&= \frac{P(E_{AB} \cap W_B)}{P(W_B)}
\end{aligned}
$$

Note that $P(E_{AB} \cap W_B) = P(W_B|E_{AB})P(E_{AB}) = 1/3$ and $P(W_B) = P(W_B|E_{BC})P(E_{BC}) + P(W_B|E_{AB})P(E_{AB}) = 1/6 + 1/3$. So the total expression evaluates to $\frac{1/3}{1/6+1/3} = 2/3$ and the chances of A are not reduced by asking after all.

Problem 4. *Show that if two events A and B are independent, then the events A and $\overline{B}$ are also independent.*

A and B independent means $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$. We have to show $\Pr(A \cap \overline{B}) = \Pr(A) \cdot \Pr(\overline{B})$. Since A is the disjoint union of $(A \cap B)$ and $(A \cap \overline{B})$ we have

$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap \overline{B})$$

and therefore

$$\begin{aligned}
\Pr(A \cap \overline{B}) &= \Pr(A) - \Pr(A \cap B) \\
&= \Pr(A) - \Pr(A)\Pr(B) \\
&= \Pr(A)(1 - \Pr(B)) \\
&= \Pr(A)\Pr(\overline{B})
\end{aligned}$$

proving that A and $\overline{B}$ are independent.

Problem 5. *Two numbers x and y are selected at random from the set $\{1, \ldots, 100\}$. Let the events A, B and C be defined as follows: $A = \{x > 50\}$, $B = \{y > 50\}$ and $C = \{x > y\}$. Are the events A and B independent? Are A and C independent?*

We of all we compute $\Pr(A) = \Pr(B) = 50/100 = 0.5$ and $\Pr(C) = \binom{100}{2} \cdot 100^{-2} = 0.495$.

Since $\Pr(A \cap B) = (50 \cdot 50)/100^2 = 0.25 = \Pr(A) \cdot \Pr(B)$, A and B are independent. However $\Pr(A \cap C) = \Pr(C) - \Pr(C \setminus A) = \binom{100}{2} \cdot 100^{-2} - \binom{50}{2} \cdot 100^{-2} = 0.3725 \neq \Pr(A) \cdot \Pr(C) = 0.247$, so A and C are not independent.

Problem 6. *A communication system transmits binary information over a channel that introduces random bit errors with probability 10^{-3}. The transmitter transmits each information bit 5 times and a decoder takes a majority vote of the received bits to decide on what the the transmitted bit was. Find the probability that the receiver will make an incorrect decision (simplify your answer to a number).*

An incorrect decision will happen if 3, 4 or 5 bits are incorrect.

$p = 10^{-3}$

$$P(incorrect) = \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + \binom{5}{5} p^5 = 9.98 \cdot 10^{-9}$$

Problem 7. *Given an experiment with a fair, six-sided die, where the die is rolled until a '6' has come up three times.*

(a) *What is the probability that at most n rolls are needed?*

First let's find the probability that *exactly* n rolls are needed. If exactly n are needed, then the final 6 must come on the nth roll, and the other two on any of the other rolls. There cannot be more than 3 6's otherwise we wouldn't need n rolls. So what is the probability of having a 6 on the nth roll, and 2 other sixes somewhere in the other $n - 1$ rolls?

$$P = \binom{n-1}{2}(\frac{1}{6})^3(\frac{5}{6})^{n-3}$$

The probability that we need at least n rolls is therefore:

$$P = \sum_{i=3}^{n} \binom{i-1}{2}(\frac{1}{6})^3(\frac{5}{6})^{i-3}$$

(b) *What if the die must come up '6' three times in a row? What is the probability that at most n rolls are needed?*

(b) *In a sequence of Bernoulli trials with success probability p, find the probability that a successes will occur before b failures.*

If we want exactly a successes will occur before b failures, then the last trial must be a failure. So we can arrange a successes and $b-1$ failures in any way before the bth failure.

$$P = \binom{a+b-1}{a} p^a(1-p)^b$$

The probability of at least a successes before b failures is:

$$P = \sum_{i=a}^{\infty} \binom{i+b-1}{i} p^i(1-p)^b$$

Problem 8. *In this problem, we will prove that there is a tournament graph with the following bizarre property: for every set of k people in the tournament, there is a certain person who has beaten all of them—this makes it hard it identify the best k players! We'll use a technique called the* probabilistic method *where you prove something exists by showing that that the probability it exists is not 0.*

Consider an n-vertex tournament that is generated randomly by assigning each edge a random (equally probable) direction.

(a) *Consider a certain set of k people. What is the probability a particular other person beats all of them?*

This person has probability 1/2 to beat each player in the set and the events are independent. By product rule the probability that beats all of them is $(1/2)^k$

(b) *What is the probability that* nobody *beats all of them?*

This is the intersection of the complements of the events considered in part (a). There are $n-k$ such events (one for each person not in the set) and they are all independent. So, the overall probability is $(1-(1/2)^k)^{n-k}$.

(c) *Using the union bound, upper bound the probability that there is some set of k people for which nobody else beats all of them?*

Using the union bound we get the upper bound

$$\binom{n}{k}(1-(1/2)^k)^{n-k}.$$

(d) *Using the identities $(1-x) < e^{-x}$ and $\binom{n}{k} < (en/k)^k$, simplify your answer to the previous problem.*

The upper bound simplifies to

$$(en/k)^k e^{-\frac{n-k}{2^k}}$$

(e) *Find the smallest n (as a function of k) that you can for which the above quantity is less than 1.*

(f) *Show that there is a tournament on n vertices (n you just found) with the claimed bizarre property.*