Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professor David R. Karger

Quiz 2
April 23, 1997

Quiz 2

- This quiz is closed book. One crib sheet is allowed.
- Put your name on the top of **every** page—**these pages will be separated for grading.**
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Do not spend too much time on any problem. Read them all through first and attack them in the order that allows you to make the most progress. I've tried to order them in increasing difficulty, but your mileage may vary.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. **Be neat**.
- This quiz ends at 9 P.M. It contains 7 problems, which vary significantly in their point value. You will get 5 extra points for circling your tutorial TA and time and clearly writing your name on *every* page.
- GOOD LUCK!

Problem	Points	Graded by
name	5	
1	40	Daniele
2	15	Alex
3	5	Daniele, Dimitri
4	15	Dimitri
5	20	David
6	10	Terri
7	10	Prof. Karger
Total	120	

Name: ______________________________

Please circle your tutorial TA: Terri David Dimitri

Please circle your tutorial time: 1 2 3 7

Problem 1. [40 points] Circle T or F for all questions, even if you have to guess.

F The relation on the natural numbers given by $\{(x, y) \mid x \geq y - 2\}$ is a poset.

F The complement of a poset is always a poset.

T Not all vertices in a 20 vertex tree are leaves.

F There is a graph with an odd number of odd degree vertices.

F Some 20-vertex tree has an Euler tour.

T A **complete** graph on $n > 3$ vertices has an Euler tour iff n is odd.

T If every vertex in a directed graph has outdegree at least 1, then it has a cycle.

F If a graph has more edges than vertices, then it is connected.

T An n-vertex graph G has a spanning forest with $n - 1$ edges iff G is connected.

T If 126 students take this quiz, then two will have scores within 1 point of each other.

F There is a surjection $f : \mathbb{N} \to \mathbb{R}$.

T If a set A contains an uncountable subset B, then A is uncountable.

T If an uncountable set A contains a countable subset B, then $A - B$ is uncountable.

F If an uncountable set A contains an uncountable subset B, then $A - B$ is countable.

F The set of functions from $\{0, 1\}$ to $\mathbb{N}$ is uncountable.

T The sum rule is a special case of inclusion-exclusion.

T The ways that four distinguishable, ordinary six-sided dice can land without any number repeated correspond to permutations from some set.

F The expansion of $(x - 1)^{120}$ has three identical nonzero coefficients.

T If $p > 5$ is prime, then $p \mid \binom{p}{p-4}$.

T The number of topological sorts of a poset is at least the product of the number of topological sorts of the upper bounds and the lower bounds of an arbitrary element.

Problem 2. [15 points] Short analyses. No need to show your work. **Be careful** to look at the summation indices.

(a) What is the product of the first n odd powers of two: $\Pi_{k=1}^{n} 2^{2k-1}$?

$$\Pi_{k=1}^{n} 2^{2k-1} = 2^{\sum_{k=1}^{n} 2k-1} = 2^{2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1} = 2^{n(n+1)-n} = 2^{n^2}$$

(b) What is $\sum_{k=0}^{\infty}(1/3)^k$?

$$\sum_{k=0}^{\infty}(1/3)^k = \lim_{k\to\infty} \frac{1-(\frac{1}{3})^{k+1}}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

(c) What is $\sum_{k=a}^{b} k$?

$$\sum_{k=a}^{b} k = \sum_{k=1}^{b} k - \sum_{k=1}^{a-1} k = \frac{b(b+1)}{2} - \frac{a(a-1)}{2}$$

(d) Give asymptotic (Θ) bounds for $\sum_{k=n/2}^{n} 1/k$.

$$sum_{k=n/2}^{n} \frac{1}{k} = \Theta(\int_{n/2}^{n} \frac{1}{k} dk)$$

$$\int_{n/2}^{n} \frac{1}{k} dk = \log(k) \mid_{n/2}^{n} = \log(\frac{n}{n/2}) = \log(2)$$

Since $\log(2)$ is a constant,

$$\sum_{k=n/2}^{n} 1/k = \Theta(1)$$

(e) Give asymptotic (Θ) bounds for $\sum_{k=n/2}^{n} k^5$.

$$\sum_{k=n/2}^{n} k^5 = \Theta(\int_{n/2}^{n} k^5 dk) = \Theta(\frac{k^6}{6} \mid_{n/2}^{n}) = \Theta(\frac{n^6}{6} - \frac{n^6}{2^6 \cdot 6}) = \Theta(n^6)$$

(f) How many subsets of $\{1, \ldots, 10\}$ are there?
The power set has size 2^{10}

(g) You open a savings account with an initial deposit of s dollars.
At the end of every year, the bank pays into the account interest of p dollars on every dollar in the account at that time. You then immediately (after the interest is paid) deposit d dollars. Give (but do not solve) a recurrence for the amount of money $M(n)$ in your account right after you make your deposit at the end of the nth year.
$M(n) = (1+p)M(n-1) + d$ with the initial condition $M(1) = s$ (or $M(0) = s$).

Problem 3. [5 points] Fill in the blank with the most accurate choice from o, Θ, and ω. Remember that o means O but not Θ, while ω means Ω but not Θ.

(a) $(\log n)^{10} = o(n)$.

Recall that one way of determining if $f(x) = o(g(x))$, is to find the limit, as $x \to \infty$, of the ratio of the functions.
If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, then $f(x) = \omega(g(x))$.
If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then $f(x) = o(g(x))$.
If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \text{constant}$, then $f(x) = \Theta(g(x))$.
Note that these are not proofs in themselves, but they tell us which one will grow faster.
Show that $(\log n)^k = o(n)$:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{(\log n)^k}{n}$$

Applying L'Hopital's rule, multiple times we get that

$$\begin{aligned}
\lim_{n\to\infty} \frac{(\log n)^k}{n} &= \lim_{n\to\infty} \frac{k(\log n)^{k-1}\frac{1}{n}}{1} \\
&= \lim_{n\to\infty} \frac{k(\log n)^{k-1}}{n} \\
&= \lim_{n\to\infty} \frac{k(k-1)(\log n)^{k-2}}{n} \\
&\cdots \\
&= \lim_{n\to\infty} \frac{k!}{n} \\
&= 0
\end{aligned}$$

So, $(\log n)^k = o(n)$, and more specifically, $(\log n)^{10} = o(n)$.

(b) $\log(n!) = \Theta(n \log n)$.

$$\lim_{n\to\infty} \frac{\log(n!)}{n\log n} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \log i}{n \log n}$$

Applying L'Hopital's rule,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \log i}{n\log n} &= \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \frac{1}{i}}{n \cdot \frac{1}{n} + \log n} \\
&= \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \frac{1}{i}}{1 + \log n}
\end{aligned}$$

$$\approx \lim_{n\to\infty} \frac{\log n}{1+\log n}$$
$$= 1$$

So, $\log(n!) = \Theta(n \log n)$.

(c) $\log n = \Theta(H_n)$.

Recall from class that $H_n = \sum_{i=1}^{n} \frac{1}{i}$ can be approximated by $\log n$, so $\log n = \Theta(H_n)$.

(d) $\binom{n}{3} = o(n^4)$

$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6} = \Theta(n^3)$$

Since $n^3 = o(n^4)$, $\binom{n}{3} = o(n^4)$.

(e) $2^n = \omega(n^2)$

This is a common problem in computer science: we want our algorithms to run in polynomial time n^k, not exponential time k^n.

By applying L'Hopital's rule twice, we get:

$$\begin{aligned}
\lim_{n\to\infty} \frac{2^n}{n^2} &= \lim_{n\to\infty} \frac{2^n \ln 2}{2n} \\
&= \lim_{n\to\infty} \frac{2^n (\ln 2)^2}{2} \\
&= \infty
\end{aligned}$$

So $2^n = \omega(n^2)$.

Problem 4. [15 points] Solve the following recurrences. You must show your derivation, but do not have to prove your answer correct.

(a) $T(n) = T(n-1) + 6T(n-2)$, $T(0) = 2$, $T(1) = 1$.
Let $T(n) = c\alpha^n$. Substituting for T we get

$$c\alpha^n = c\alpha^{n-1} + 6c\alpha^{n-2}$$

$$\alpha^2 - \alpha - 6 = 0$$

$$(\alpha - 3)(\alpha + 2) = 0$$

$$\alpha = 3, -2$$

So, $T(n) = a3^n + b(-2)^n$.
Using our initial conditions:

$$T(0) = 2 = a + b$$

$$T(1) = 1 = 3a - 2b$$

From these we get, $a = 1$ and $b = 1$, so
So, $T(n) = 3^n + (-2)^n$.

(b) $T(n) = 4(T(n-1) - T(n-2))$, $T(0) = 1$, $T(1) = 0$.
Let $T(n) = c\alpha^n$. Substituting for T we get

$$c\alpha^n - 4c\alpha^{n-1} + 4c\alpha^{n-2} = 0$$

$$\alpha^2 - 4\alpha + 4 = 0$$

$$(\alpha - 2)^2 = 0$$

$$\alpha = 2$$

Since we have a double root, we choose $T(n) = a2^n + bn2^n$. For $T(0)$:

$$T(0) = 1 = a$$

$$T(1) = 0 = 2a + 2b$$

From these we get, $a = 1$ and $b = -1$, so $T(n) = 2^n - n2^n$.

(c) $T(n) = 9T(n/3) + n^3$, $T(1) = 1$, where n is a power of 3 (an asymptotic bound is sufficient here).

$$T(n) = 9T(\frac{n}{3}) + n^3$$

Let $n = 3^k$

$$T(3^k) = 9T(3^{k-1}) + 3^{3k}$$

Let $S(k) = T(3^k)$, $S(0) = 1$.

$$
\begin{aligned}
S(k) &= 9S(k-1) + 3^{3k} \\
&= 3^{3k} + 9(3^{3(k-1)} + 9(3^{3(k-2)} + \cdots + 9S(0))) \\
&= 3^{3k} + 3^{3k-1} + 3^{3k-2} + \cdots + 3^{2k+1} + 9^k S(0) \\
&= \sum_{i=2k+1}^{3k} 3^i + 9^k \\
&= \sum_{i=1}^{3k} 3^i - \sum_{i=1}^{2k} 3^i + 9^k \\
&= \frac{1-3^{3k+1}}{1-3} - \frac{1-3^{2k}}{1-3} + 9^k \\
&= \frac{3^{3k+1} - 3^{2k}}{2} + 9^k \\
&= \frac{3 \cdot 27^k - 9^k}{2} + 9^k
\end{aligned}
$$

Now, substitute back for $k = \log_3 n$, and $T(3^k) = S(k)$.

$$
\begin{aligned}
T(3^k) = S(k) &= \frac{3 \cdot 27^{(\log_3 n)} + 9^{(\log_3 n)}}{2} \\
T(n) &= \frac{3n^3 + n^2}{2} \\
&= \Theta(n^3)
\end{aligned}
$$

<u>comments:</u>

- Some people tried to solve the linear difference equation as a recurrence and vice-versa.
- Some people found the correct substitution of variables for part (b) but the kept trying to solve the problem by expanding the recurrence instead of solving the difference equation.

Problem 5. [20 points] Consider the relation R on $\{2,\ldots,100\}$ defined by aRb if $a \mid b$.

(a) Prove that R is a poset

Let $S = \{2,\ldots,100\}$. Posets are reflexive, antisymmetric, and transitive. Recall that

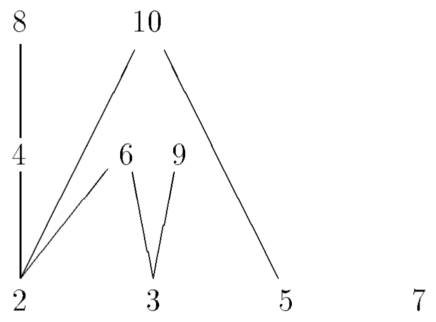
- reflexive means $\forall a \in S, (a,a) \in R$,
- antisymmetric means that $(a,b) \in R \wedge (b,a) \in R \rightarrow a = b$, and
- transitive means that $(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$.)

R is reflexive: $\forall a \in \{2,\ldots,100\}$, $a \mid a$.

R is antisymmetric: $(a \mid b) \wedge (b \mid a)$ can only be true if $a = b$ since if $a \mid b$ then $b = ak_1$, $k \in \mathbb{N}, k \geq 1$, but if $b \mid a$, then, $a = bk_2$, so $b = b \cdot k_1 \cdot k_2$, but since k_1 and k_2 are integers, $k_1 = k_2 = 1$, so $a = b$.

Transitivity: If $(a \mid b) \wedge (b \mid c)$, then $b = ak_1$, and $c = bk_2$, so $c = ak_1k_2 = ak_3$, so $a \mid c$, and $(a,c) \in R$.

(b) Draw the Hasse diagram for R on the numbers $2,\ldots,10$.



(c) What (1 word) are the minimal elements of this particular poset called?

primes

(d) What is the greatest lower bound of two elements in this poset called?

GCD

(e) What is the longest chain in R (over $\{2,\ldots,100\}$)?

$\lfloor \log_2 100 \rfloor = 5$ $(2 \rightarrow 4 \rightarrow 8 \rightarrow 16 \rightarrow 32 \rightarrow 64)$

(f) What does the previous part imply about the largest antichain? Why?

Recall that Dilworth's theorem says:

$\forall t$, every poset with n elements must have a chain of size $> t$ or an antichain of size $\geq \frac{n}{t}$.

So, since there is a chain of size 6, there must be an antichain of size $\geq \left\lfloor \frac{99}{5} \right\rfloor = 19$.

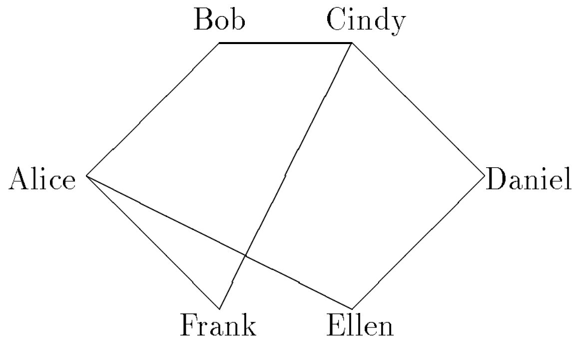
(g) Prove that $\{51,\ldots,100\}$ is an antichain and that no set strictly containing it is an antichain.

$Q = \{51,\cdots,100\}$ is an antichain since if any distinct $s_1 \in Q$ and $s_2 \in Q$, had $s_1 | s_2$, then $s_2 = ks_1$ but the minimum k is 2 if $s_1 \neq s_2$, so $s_2 = 2s_1$, but $2s$ for any $s \in Q$ is between 102 and 200.

No set containing Q can be an antichain, since it would contain some number, n where $2 \leq n \leq 50$, but we can find some integer k such that $51 \leq kn \leq 100$ for all $2 \leq n \leq 50$.

Problem 6. [10 points] Alice, Bob, Cindy, Daniel, Ellen, and Frank work at MIT. Alice works Monday, Tuesday, and Saturday. Bob works Tuesday and Wednesday. Cindy works Sunday, Wednesday, and Thursday. Daniel works Thursday and Friday. Ellen works Monday and Friday. Frank works Saturday and Sunday. We want to assign each worker a single parking space at MIT such that at most one person uses each parking space on a given day. Of course, people who work different days can share a parking space.

(a) Draw a conflict graph capturing the constraints on who can share parking spaces.



(b) Give an assignment of people to spaces that uses the fewest possible parking spaces.

There are a variety of solutions, all involving three-colors (or parking spots), one is:
Alice and Cindy
Bob, Frank, and Daniel
Ellen

(c) Prove that you cannot do better.

To do better, we'd have to be able to color the graph with two colors, but this would mean the graph is a bipartite. Since it has an odd cycle (ABCDE), it cannot be a bipartite, therefore 3 colors is the best we can do.

Problem 7. [10 points] For a tree T, let $L(T)$ denote the number of leaves in T and let $H(T)$ be the number of vertices whose degree **exceeds** 2. Prove by induction on the number of vertices in T that $L(T) \geq H(T)$.

Proof by induction the number of vertices, n, in the tree.

We'll start with a base case of 2 vertices since otherwise we need to be careful about degree 0 vertices.

Base case: 2 vertices. Graph is a single edge.
$L(T_1) = 2, H(T_1) = 0$, and $L(T_1) \geq H(T_1)$.

Inductive hypothesis: $P(n)$: $L(T) \geq H(T)$ for any tree of size n or fewer vertices.

Show $P(n) \rightarrow P(n+1)$
Given a tree, T_{n+1}, of size $n+1$ vertices. There must exist a leaf (by a theorem shown in class), so remove a leaf. The new tree, T_n, must have $L(T_n) \geq H(T_n)$ by the inductive hypothesis. Now, by removing the vertex from T_{n+1}, the node went from a state of having X edges to Y edges:

- >3 edges to ≥ 3 edges

  $H(T_{n+1}) = H(T_n), L(T_{n+1}) = L(T_n) + 1$, so

  $$H(T_{n+1}) = H(T_n) \leq L(T_n) = L(T_{n+1}) - 1 \leq L(T_{n+1})$$

- 3 edges to 2 edges

  $H(T_{n+1}) = H(T_n) + 1$ and $L(T_{n+1}) = L(T_n) + 1$ so

  $$H(T_{n+1}) = H(T_n) + 1 \leq L(T_n) + 1 = L(T_{n+1})$$

  so $H(T_{n+1}) \leq L(T_{n+1})$.

- 2 edges to 1 edge

  $H(T_{n+1}) = H(T_n)$, and $L(T_{n+1}) = L(T_n) + 1$, so

  $$H(T_{n+1}) = H(T_n) \leq L(T_n) = L(T_{n+1}) - 1 \leq L(T)$$

- 1 edge to no edges
  n must be 2 in this case. $H(T_2) = 0 \leq L(T_2) = 2$, so $0 = H(T_2) \leq L(T_2) = 2$.

Comments:

1. it is easier for this problem to remove a leaf than an internal edge. This is often the case.
2. A lot of people didn't notice that when you remove a leaf, you might create a new leaf.