Problem Set 10 Solutions

Problem 1. Binomial Theorem

(a) What is the coefficient of x^k in $(x^2 + \frac{1}{x^2})^{50}$?
We know that $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$. Applying this, we find that

$$\begin{aligned}
(x^2 + \frac{1}{x^2})^{50} &= \sum_{i=0}^{50} \binom{50}{i} (x^2)^i (\frac{1}{x^2})^{50-i} \\
&= \sum_{i=0}^{50} \binom{50}{i} (x^2)^{i-(50-i)} \\
&= \sum_{i=0}^{50} \binom{50}{i} x^{4i-100}
\end{aligned}$$

Since we're looking for the coefficient on the x^k term, let $k = 4i - 100$, so $i = k/4 + 25$. The coefficient is therefore, $\binom{50}{k/4+25}$ when k is a multiple of 4.

(b) Let $a, n \in \mathbb{N}$. Show that the expression

$$\frac{1}{\sqrt{a}}[(1+\sqrt{a})^n - (1-\sqrt{a})^n]$$

is an integer.

$$\begin{aligned}
\frac{1}{\sqrt{a}}[(1+\sqrt{a})^n - (1-\sqrt{a})^n] &= \frac{1}{\sqrt{a}}(\sum_{i=0}^{n} \binom{n}{i} (\sqrt{a})^i - \sum_{i=0}^{n} \binom{n}{i} (-\sqrt{a})^i) \\
&= \sum_{i=0}^{n} \binom{n}{i} (\sqrt{a})^{i-1} + \sum_{i=0}^{n} \binom{n}{i} (-\sqrt{a})^{i-1} \\
&= \sum_{i=0}^{n} \binom{n}{i} (\sqrt{a})^{i-1} + \sum_{i=0}^{n} \binom{n}{i} (-1)^{i-1} (\sqrt{a})^{i-1} \\
&= \sum_{i=0}^{n} \binom{n}{i} (\sqrt{a})^{i-1} (1 + (-1)^{i-1}) \\
&= \sum_{i=0}^{n} \binom{n}{i} (a^{\frac{i-1}{2}} (1 + (-1)^{i-1}) \\
&= \sum_{i=0}^{n} \binom{n}{i} (a^{\frac{i-1}{2}}) (1 + (-1)^{i-1})
\end{aligned}$$

If i is even , then $(-1)^{i-1}$ is -1, and $1 - 1 = 0$, so all terms where i is even are 0.
If i is odd, (say $i = 2k+1$), then $(-1)^{i-1}$ is 1, and $1 + 1 = 2$, so all such terms have a non-zero, integral coefficient, and $a^{\frac{i-1}{2}} = a^k$ is an integer since a is an integer.

Problem 2. Inclusion-Exclusion

(a) Prove the principle of inclusion-exclusion using mathematical induction.

The base case is $n = 2$, for which we already know the formula to be valid. Assume that the formula is true for n sets. We will show that it is true for $n + 1$ sets.

$|A_1 \cup \ldots \cup A_{n+1}| = \sum_{i<n} |A_i| + |A_n \cup A_{n+1}| - \sum_{i<j<n} |A_i \cap A_j|$
$- \sum_{i<n} |A_i \cap (A_n \cup A_{n+1})| + \ldots + (-1)^n |A_1 \cap \ldots \cap A_{n-1} \cap (A_n \cup A_{n+1})|$.

Next we apply the distributive law to each term involving $A_n \cup A_{n+1}$.

$$\sum |(A_{i_1} \cap \ldots \cap A_{i_m}) \cap (A_n \cup A_{n+1})| = \sum |(A_{i_1} \cap \ldots \cap A_{i_m} \cap A_n) \cup (A_{i_1} \cap \ldots \cap A_{i_m} \cap A_{n+1})|.$$

Now we apply the base case to rewrite each of these terms as

$$\sum |(A_{i_1} \cap \ldots \cap A_{i_m} \cap A_n| + \sum |(A_{i_1} \cap \ldots \cap A_{i_m} \cap A_{n+1}| - \sum |A_{i_1} \cap \ldots \cap A_{i_m} \cap A_n \cap A_{n+1}|,$$

which gives us precisely the summation we want.

(b) An integer is called **square free** if it is not divisible by the square of a positive integer greater than 1. Find the number of square-free positive integers less than 100.

$$99 - \lfloor 99/2^2 \rfloor - \lfloor 99/3^2 \rfloor - \lfloor 99/5^2 \rfloor - \lfloor 99/7^2 \rfloor + \lfloor 99/(2^2 3^2) \rfloor = 61.$$

Problem 3. Use a tree diagram to determine the number of ways that the world series can occur, where the first team to win four games out of 7 wins the series.

Problem 4. How many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people that includes the bride and groom if

(a) The bride must be in the picture?

There are 6 ways to position the bride, $\binom{9}{5}$ ways to choose the other people to appear, and 5! ways to arrange them. This gives a total of $6 \cdot 5! \cdot \binom{9}{5} = 6! \cdot \binom{9}{5}$ ways.

(b) The bride and the groom must be in the picture?

There are 6 ways to place the bride and 5 to place the groom. There are $\binom{8}{4}$ ways to choose the people to appear and 4! ways to arrange them. Total : $6! \cdot \binom{8}{4}$ ways.

(c) Exactly one of the bride and groom must be in the picture?

2 ways to choose between bride or groom, 6 ways to place whichever one is chosen, and $5! \cdot \binom{8}{5}$ to choose and arrange the remaining people. Total : $2 \cdot 6! \cdot \binom{8}{5}$ ways.

Problem 5. Seven women and nine men are on the faculty of a school's math department. How many ways are there to select a committee of 5 members if

(a) at least 1 woman must be on the committee?

We need to count all possible combinations of people such that there is at least one woman in every combination, but we must remember not to count any combinations mutiple times.

We can have committees with

1 woman, 4 men: $\binom{7}{1}\binom{9}{4}$
2 women, 3 men: $\binom{7}{2}\binom{9}{3}$
3 women, 2 men: $\binom{7}{3}\binom{9}{2}$
4 women, 1 man: $\binom{7}{4}\binom{9}{1}$
5 women, 0 men: $\binom{7}{5}\binom{9}{0}$

So there are $\binom{7}{1}\binom{9}{4}+\binom{7}{2}\binom{9}{3}+\binom{7}{3}\binom{9}{2}+\binom{7}{4}\binom{9}{1}+\binom{7}{5}\binom{9}{0} = 4242$ different possibilities for committees.

Another way to solve this problem is to say that there are $\binom{16}{5}$ different committees, and $\binom{9}{5}$ committees of just men. So there are $\binom{16}{5} - \binom{9}{5}$ different possibilites for committees.

5 women, 0 men: $\binom{7}{5}\binom{9}{0}$

Note that $7 * \binom{15}{4}$ is not a correct answer. The product rule only applies when you are choosing two separate sets of items which are independent of each other. In this case, a woman is chosen and then the remainder of the committee is chosen, but since there may be women in the last set of 4 chosen, we will be double counting. For example, if A and B are women, and C, D, and E are men, then we will count $A, \{BCDE\}$ as well as $B, \{ACDE\}$.

(b) at least 1 man and 1 woman must be on the committee? We can have committees with

1 woman, 4 men: $\binom{7}{1}\binom{9}{4}$
2 women, 3 men: $\binom{7}{2}\binom{9}{3}$
3 women, 2 men: $\binom{7}{3}\binom{9}{2}$
4 women, 1 man: $\binom{7}{4}\binom{9}{1}$

So there are $\binom{7}{1}\binom{9}{4}+\binom{7}{2}\binom{9}{3}+\binom{7}{3}\binom{9}{2}+\binom{7}{4}\binom{9}{1} = 4221$ different possibilities for committees.

Again, this is also equal to the number of different committees, minus the number of committees with no men, and minus the number of committees with no women: $\binom{16}{5} - \binom{9}{5} - \binom{7}{5}$

(c) there must be more men than women on the committee?

For there to be more men than women, we must have 3 or more men on the committee.

5 men, 0 women: $\binom{9}{5}\binom{7}{0}$
4 men, 1 woman $\binom{9}{4}\binom{7}{1}$
3 men, 2 women: $\binom{9}{3}\binom{7}{2}$

$\binom{9}{5}\binom{7}{0}+\binom{9}{4}\binom{7}{1}+\binom{9}{3}\binom{7}{2} = 2772$

Problem 6. Five ordinary distinguishable six-sided dice are rolled.

(a) How many possible ways can they land?

Five ordinary distinguishable six-sided dice can land in $6^5 = 7776$ ways.

(b) How many possible ways where the sum of the first two equals the sum of the other three?

Fix a sum $k, 3 \le k \le 12$. There are $\binom{k-1}{k-2}$ ways to form this sum with two positive integers and $\binom{k-1}{k-3}$ ways to form it with 3. From these we must subtract the cases where the constraint $i \le 6$

is violated for die i. This is equivalent to solving the equations $x_1 + x_2 = k - 6$, for $x_i > 0$ and $x_1 + x_2 + x_3 = k - 6$, for $x_i > 0$. So, there are $\binom{k-7}{k-8}$ solutions to the first equation and $\binom{k-7}{k-9}$ to the second equation.

So there are

$$\sum_{k=3}^{12} \left(\binom{k-1}{k-2} - \binom{k-7}{k-8} \right) \cdot \left(\binom{k-1}{k-3} - \binom{k-7}{k-9} \right)$$

where it is understood that the expressions are 0 when not defined.

(c) How many possible ways where there are an even number of dice coming up even?

For each even number k, there are $\binom{5}{k}$ ways to choose the dice, 3^k ways to choose the numbers on them, and 3^{5-k} ways to choose the odd numbers in the remaining dice. So there are, $\binom{5}{0} \cdot 3^0 \cdot 3^5 \binom{5}{2} \cdot 3^2 \cdot 3^3 + \binom{5}{4} \cdot 3^4 \cdot 3$ ways.

(d) How many ways where the dice have a full house (that is, two of one number plus three of a different number)?

To get a full house, there are $\binom{5}{2}$ ways to pick which dice are the pair and which are the triple, and 6×5 ways to choose the values of the pair and triple: $\binom{5}{2} \times 6 \times 5 = 300$.

(e) How many ways such that no pair shows?

If they land so that no pair shows up, there are $6 \times 5 \times 4 \times 3 \times 2 = 720$ possibilities.

(f) How many ways with no two dice showing consecutive numbers?

Note this is not asking for "no two consecutive dice showing consecutive numbers" but just no two dice showing consecutive numbers.

There are only a few combinations of values that can appear on the dice. These are $\{1, 3, 5\}$, $\{2, 4, 6\}$, $\{1, 3, 6\}$, $\{1, 4, 6\}$, and $\{2, 5\}$. (Note that combinations like $\{1, 5\}$ and $\{3, 6\}$ are subsets of one of the above sets, and will be counted when we count the permutations of these sets.)

Let's examine each of these possible combinations of values, counting how many permutations we can get (with values repeating), and then subtract off any permutations that we've already counted.

- $\{1, 3, 5\}$:
  3^5 possible permutations (each die can have the value 1, 3, or 5).
- $\{2, 4, 6\}$:
  3^5 (no overlap with $\{1, 3, 5\}$).
- $\{1, 3, 6\}$:
  $3^5 - 3$ (11111 and 33333 and 66666 have already been counted).
- $\{1, 4, 6\}$:
  $3^5 - 2^5 - 1$ (All combinations of just 1 and 6 were counted in $\{1, 3, 6\}$. Also 44444 has already been counted).
- $\{2, 5\}$:
  $2^5 - 2$ (22222 and 55555 have already been counted).

So, all in all we have: $3^5 + 3^5 + (3^5 - 3) + (3^5 - 2^5 - 1) + (2^5 - 2) = 966$

Problem 7. Prove that

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

(a) algebraically.

$$\begin{aligned}\binom{n}{r}\binom{r}{k} &= \frac{n!}{r!(n-r)!}\frac{r!}{k!(r-k)!} \\ &= \frac{n!}{(n-r)!(r-k)!k!} \\ &= \frac{n!(n-k)!}{(n-r)!(r-k)!k!(n-k)!} \\ &= \frac{n!}{k!(n-k)!}\frac{(n-k)!}{(n-k-(r-k))!(r-k)!} \\ &= \binom{n}{k}\binom{n-k}{r-k}\end{aligned}$$

(b) using a combinatorial argument.

Lets say that we have n dogs, and we wish to put some of them in two kennels. We can fit k dogs in one of the kennels and $r-k$ dogs in the other. There are two ways we can choose which dogs to put in which kennels:

1. From the n dogs, choose r dogs which will be put into kennels: $\binom{n}{r}$. Then, from the r dogs, choose k dogs which will be put into the first kennel, and put the remaining $r-k$ into the second kennel. $\binom{r}{k}$. This gives us $\binom{n}{r}\binom{r}{k}$.

2. From the n dogs, choose k dogs to be put into the first kennel: $\binom{n}{k}$. Then, from the remaining $n-k$ dogs, choose $r-k$ dogs which will be put into the second kennel: $\binom{n-k}{r-k}$. This gives us $\binom{n}{k}\binom{n-k}{r-k}$.

Problem 8. Give a combinatorial proof that

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

Hint: count the number of ways to build a committee of n people from among n math and n computer science professors, assuming the chair is a math prof. Consider two committees different if they have different chairs.

There are n ways to choose a chair from the mathematicians, and then $\binom{2n-1}{n-1}$ ways to choose the rest of the members.

On the other hand, for a fixed number of k mathematicians in the committee, there are $\binom{n}{k}$ ways to choose the mathematicians, k ways to choose the chair among the mathematicians and, $\binom{n}{n-k} = \binom{n}{k}$ ways to choose the computer scientists. So there are $\sum_{k=1}^{n} k\binom{n}{k}^2$ ways to form the committee.

Problem 9. How many relations are there on the set $\{1,\ldots,n\}$? How many reflexive ones? How many symmetric ones? How many that are reflexive and symmetric?

Let $S = \{1,\ldots,n\}$, and R be a relation on $S \times S$.

A relation is a set of ordered pairs, how many distinct ordered pairs do we have to choose from? $|S\times S| = n^2$, so there are n^2 ordered pairs.

- How many relations are there on the set $\{1, \ldots, n\}$?

  The number of distinct relations on S is the number of distinct subsets we can make from $S \times S$. This is the power set of $S \times S$ which has size $2^{|S \times S|} = 2^{n^2}$.

- How many reflexive ones?

  For a relation to be reflexive, $(x, x) \in R$ for all $x \in S$. There are n reflexive pairs, which leaves $n^2 - n$ ordered pairs in $S \times S$. The number of possible reflexive relations is therefore the size of the power set of the remaining pairs, which has size 2^{n^2-n}. So there are 2^{n^2-n} reflexive relations.

- How many symmetric ones?

  For a relation to be symmetric, for any (a, b) in R where $a \neq b$, (b, a) must also be in R.
  Since $(x, x) \in R \rightarrow (x, x) \in R$, all the reflexive ordered pairs are symmetric with themselves. This means that there are n reflexive ordered pairs.
  For all, $a \neq b$, if we select (a, b) to be in R, it comes with a partner, (b, a), So there are $(n^2 - n)/2$ sets of non-reflexive, symmetric ordered pairs ((a, b) and (b, a) with $a \neq b$).
  So the number of ways of ordering these into relations is the power set of $(n^2 - n)/2 + n = (n^2 + n)/2$ items. The size of this power set is $2^{(n^2+n)/2}$.

- How many that are reflexive and symmetric?

  There are n reflexive pairs, which must go into any relation, and $n^2 - n$ other ordered pairs. These other pairs (a, b) all have $a \neq b$, so there are $(n^2 - n)/2$ symmetric pairs of ordered pairs that can be added to any reflexive relation. Since the power set of the symmetric pairs of ordered pairs is $2^{(n^2-n)/2}$, that is how many reflexive, symmetric relations there can be.

Problem 10. How many graphs are there on 4 distinguishable vertices that do not contain any cycles of length exactly 3?

There are $\binom{4}{2} = 6$ possible edges, so there are $2^6 = 64$ graphs of 4 vertices. Now we use inclusion-exclusion to compute the number of graphs that contain cycles of length 3.

Let A_i denote the set of graphs that have a cycle facing vertex i. For example, A_1 is the set of graphs containing the cycle $2, 3, 4, 2$. There are 4 such sets. Each of the sets has 2^3 elements (all the ways to connect a vertex i to the other 3). So, $|A_1| + |A_2| + |A_3| + |A_4| = 4 \cdot 8 = 32$. From this we subtract the number of graphs having two or more cycles of length 3. There are $\binom{4}{2} = 6$ ways to pick the vertices and 2 ways to complete each of the resulting graphs (by either including or excluding the remaining edge). So there are $2 \cdot 6 = 12$ elements in the sum of the pairwise intersections. Then we consider the set consisting of intersection of three graphs containing cycles. There are $\binom{4}{3} = 4$ ways to choose the vertices, and just one graph of this type (K_4). Finally we consider the intersection of the graphs containing all four cycles which consists of the single element K_4. In total there are $32 - 12 + 4 - 1 = 23$ graphs containing cycles of length 3, so there are $64 - 23 = 41$ graphs without length 3 cycles.

Optional Problem 11. Evaluate the following summation:

$$\sum_{k=0}^{m} \binom{m}{k} / \binom{n}{k}$$

By fixing n and experimenting with a few values for m a pattern arises. we guess the identity

$$\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n}{k}} = 1 + \frac{m}{n-m+1}$$

which we will verify by induction on m.

The base case for $m = 0$ is trivial.

Assume that the above expression holds for some m. We will show it for $m + 1$.

$$\begin{aligned}
&\sum_{k=0}^{m+1} \frac{\binom{m+1}{k}}{\binom{n}{k}} \\
&\qquad = \frac{\binom{m+1}{0}}{\binom{n}{0}} + \sum_{k=1}^{m+1} \frac{\binom{m+1}{k}}{\binom{n}{k}} \\
&\qquad = 1 + \sum_{k=1}^{m+1} \frac{\frac{m+1}{k}}{\frac{n}{k}} \frac{\binom{m}{k-1}}{\binom{n-1}{k-1}} \\
&\qquad = 1 + \frac{m+1}{n} \sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n-1}{k}} \\
&\qquad = 1 + \frac{m+1}{n} \left(1 + \frac{m}{n-m}\right) \\
&\qquad = 1 + \frac{m+1}{n-m}
\end{aligned}$$

which is the desired expression.