Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professor David R. Karger

Handout 31
May 6, 1996

Problem Set 12

Due: Never.

This homework is **entirely optional.** But it will be very good practice for the final. If you hand it in Tuesday May 12, it will be corrected (but the grade won't count for anything).

Problem 1. The Three-Cornered Duel.

A, B, and C are to fight a three-cornered pistol duel. They all know that A's chance of hitting the target is 0.3, C's is 0.5 and B never misses. They are to shoot in the sequence A, B, C and fire at their choice of target until one person is left standing. Show that A is better off missing deliberately the first shot than attempting to hit either B or C.

Problem 2. If you repeatedly flip a fair coin, what is the probability you see the sequence HTT before you see the sequence HHT? (Hint: the answer is **not** 1/2).

Problem 3. The Prisoner's Dilemma.

Three Prisoners A,B, and C have applied for parole. The parole board has decided to release two of the three, and the prisoners know this but not which two. A warder friend of prisoner A knows who are to be released. A realizes that it would be unethical to ask if he is going to be released, but thinks of asking for the name of *one prisoner other than himself* who is to be released. Before he asks his chances are 2/3, but he thinks that if he asks and the warder says "B will be released", his own chances will become 1/2 since either A or C will be the only remaining candidates. So A decides not to reduce his chances asking. Is A's argument correct? Show what his chances are after asking.

Problem 4. Show that if two events A and B are independent, then the events A and $\overline{B}$ are also independent.

Problem 5. Two numbers x and y are selected at random from the set $\{1, \ldots, 100\}$. Let the events A, B and Cbe defined as follows: $A = \{x > 50\}$, $B = \{y > 50\}$ and $C = \{x > y\}$. Are the events A and B independent? Are A and C independent?

Problem 6. A communication system transmits binary information over a channel that introduces random bit errors with probability 10^{-3}. The transmitter transmits each information bit 5 times and a decoder takes a majority vote of the received bits to decide on what

the the transmitted bit was. Find the probability that the receiver will make an incorrect decision (simplify your answer to a number).

Problem 7. Given an experiment with a fair, six-sided die, where the die is rolled until a '6' has come up three times.

(a) What is the probability that at most n rolls are needed?

(b) What if the die must come up '6' three times in a row? What is the probability that at most n rolls are needed?

(b) In a sequence of Bernoulli trials with success probability p, find the probability that a successes will occur before b failures.

Problem 8. In this problem, we will prove that there is a tournament graph with the following bizarre property: for every set of k people in the tournament, there is a certain person who has beaten all of them—this makes it hard it identify the best k players! We'll use a technique called the *probabilistic method* where you prove something exists by showing that that the probability it exists is not 0.

Consider an n-vertex tournament that is generated randomly by assigning each edge a random (equally probable) direction.

(a) Consider a certain set of k people. What is the probability a particular other person beats all of them?

(b) What is the probability that *nobody* beats all of them?

(c) Using the union bound, upper bound the probability that there is some set of k people for which nobody else beats all of them?

(d) . Using the identities $(1-x) < e^{-x}$ and $\binom{n}{k} < (en/k)^k$, simplify your answer to the previous problem.

(e) Find the smallest n (as a function of k) that you can for which the above quantity is less than 1.

(f) Show that there is a tournament on n vertices (n you just found) with the claimed bizarre property.