Massachusetts Institute of Technology — Handout 30
6.042J/18.062J: Mathematics for Computer Science — April 29, 1997
Professor David R. Karger

Problem Set 11

Due: May 6, 1997.

Reading: Rosen 4.4-4.5 up to "independence," 4.6.

In the problems below, try to simplify your answers as much as possible. There are a lot of questions, but many of them can be answered with one equation and a brief explanation.

Problem 1. Several dice and an urn.

(a) When rolling a pair of fair dice, what is the probability of getting a sum of two on the dice? a sum of seven on the dice? a sum of eleven on the dice?

(b) Pepys wrote Newton to ask which of three events is more likely: that a person get (i) at least one six when 6 dice are rolled, (ii) at least two sixes when 12 diced are rolled, or (iii) at least three sixes when 18 dice are rolled. What is the answer?

(c) A six-sided die is tossed 12 times. What is the probability of obtaining exactly 2 of each number from 1 to 6?

(d) An urn contains five balls numbered $1, 2, 3, 4, 5$. Suppose we draw three balls with replacement. What is the probability that all three balls are different?

Problem 2. When playing *Let's Make a Deal* in class, I decided to "speed things up" (give contestants a better chance of winning) by adding a 4th box and, after your choice, revealing 2 that don't have the prize. Draw the tree of outcomes for the "always switch" rule when 4 boxes are involved. What is the probability that you win?

Problem 3. Consider a best 3 out of 5 playoff. If team 1 has a probability 1/2 of winning each game that it plays, what is the most likely (that is, highest probability) number of games in the playoff? What if the first team's win probability is 9/10?

Problem 4. Suppose you are dealt a random poker hand of 5 cards from a deck of 52 cards. Compute the probability that you are dealt

(a) Four of a kind (four cards with the same value)

(b) A flush (5 cards from the same suit)

(c) Two pair (two pairs, where each pair is two cards with the same value, but the two pairs have different values and the remaining card does not have either value)

Problem 5. A batch of 50 items contains 10 defective items. Suppose 5 of these items are selected at random (without replacement) and tested.

(a) What is the probability that exactly 4 of the items tested are defective?

(b) What is the most likely number of defective items in the sample?

Problem 6. Suppose that I randomly permute the letters in MISSISSIPPI. What is the probability that all the S's end up next to each other?

Problem 7. Prove that if $A_1, \ldots, A_n$ is a partition of the sample space, then

$$\Pr[B] = \sum_{i=1}^{n} \Pr[B \mid A_i] \cdot \Pr[A_i]$$

Problem 8. A witness to a crime says that the criminal had a beard. We know from experience that if a criminal has a beard, witnesses will say so with probability 0.8, and if the criminal does not have a beard, witnesses will say that he does with probability 0.12. If 15% of the population has beards, what is the probability that a random criminal has a beard if the witness says he does?

Optional Problem 9. Suppose you are playing the *Let's Make a Deal* game, but you happen to know that the prize is going to be behind doors 1, 2, or 3 with probabilities $p_1 \geq p_2 \geq p_3$, respectively. How should you play? What are your chances of winning? Can you prove your strategy is optimal?

Optional Problem 10. Let f be a function $f\colon \{0,1\}^n \to \{0,1\}$. We think of f as an electoral system in which if people vote $X_1, \ldots, X_n$, $f(X_1, \ldots, X_n)$ wins the elections.

A natural requirement on f is monotonicity:

$$\forall i. X_i \leq Y_i \Rightarrow f(X_1, \ldots, X_n) \leq f(Y_1, \ldots, Y_n),$$

i.e. if some people change their vote from V to $1 - V$, this can only change the result of the elections from V to $1 - V$.

We measure the quality of an electoral system f by the probability that a randomly selected person R can change the outcome of the elections by changing his vote, when all other people vote independently and uniformly at random.

What is the the "best" electoral system, i.e., which monotone function f maximizes the probability that

$$f(X_1, \ldots, X_R, \ldots, X_n) \neq f(X_1, \ldots, 1 - X_R, \ldots, X_n)$$

when $X_1, \ldots, X_n$ are independent uniform binary random variables and R is an independent uniform random variable with values in $\{1, \ldots, n\}$?