Problem Set 10

Due: April 29, 1997.

Reading: Rosen 4.1–4.3, 5.4-5.5

Warning: Many of these problems ask "how many." You should always justify your answers using the counting principles given in class.

Problem 1. Binomial Theorem

(a) What is the coefficient of x^k in $(x^2 + \frac{1}{x^2})^{50}$?

(b) Let $a, n \in \mathbb{N}$. Show that the expression

$$\frac{1}{\sqrt{a}}[(1+\sqrt{a})^n - (1-\sqrt{a})^n]$$

is an integer.

Problem 2. Inclusion-Exclusion

(a) Prove the principle of inclusion-exclusion using mathematical induction.

(b) An integer is called **square free** if it is not divisible by the square of a positive integer greater than 1. Find the number of square-free positive integers less than 100.

Problem 3. Use a tree diagram to determine the number of ways that the world series can occur, where the first team to win four games out of 7 wins the series.

Problem 4. How many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people that includes the bride and groom if

(a) The bride must be in the picture?

(b) The bride and the groom must be in the picture?

(c) Exactly one of the bride and groom must be in the picture?

Problem 5. Seven women and nine men are on the faculty of a school's math department. How many ways are there to select a committee of 5 members if

(a) at least 1 woman must be on the committee?

(b) at least 1 man and 1 woman must be on the committee?

(c) there must be more men than women on the committee?

Problem 6. Five ordinary distinguishable six-sided dice are rolled.

(a) How many possible ways can they land?

(b) How many possible ways where the sum of the first two equals the sum of the other three?

(c) How many possible ways where there are an even number of dice coming up even?

(d) How many ways where the dice have a full house (that is, two of one number plus three of a different number)?

(e) How many ways such that no pair shows?

(f) How many ways with no two dice showing consecutive numbers?

Problem 7. Prove that

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

(a) algebraically.

(b) using a combinatorial argument.

Problem 8. Give a combinatorial proof that

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

Hint: count the number of ways to build a committee of n people from among n math and n computer science professors, assuming the chair is a math prof. Consider two committees different if they have different chairs.

Problem 9. How many relations are there on the set $\{1, \ldots, n\}$? How many reflexive ones? How many symmetric ones? How many that are reflexive and symmetric?

Problem 10. How many graphs are there on 4 distinguishable vertices that do not contain any cycles of length exactly 3?

Optional Problem 11. Evaluate the following summation:

$$\sum_{k=0}^{m} \binom{m}{k} / \binom{n}{k}$$