Review for Quiz 2

Problem 1. Prove that among any set of 150 natural numbers, there must be three numbers, a, b, and c, such that all the pairwise differences, $a-b, a-c, b-c, b-a, c-a, c-b$, are multiples of 70.

Problem 2. For each of the following sets, decide whether the set is countable or uncountable and prove your answer. (If an answer to one part of the question follows from an answer you have already given to another part, then you need only show that is follows – you don't need to prove the second part from scratch.)

(a) The set E of all the even natural numbers.

(b) The set of all subsets of E.

(c) The set of all finite subsets of E.

(d) The set of all even-size finite subsets of E.

Problem 3. The CS department has just completed a revamping of its curriculum. The following 10 courses are now required: 6.001, 6.002, 6.003, ..., 6.009, and 6.042.
In addition, for every i, j, 6.00i is a prerequisite of 6.00j or 6.0j, if $i < j$ and $gcd(i, j) > 1$.

(a) Write down all the elements of the *prerequisite* relation.

(b) State whether this relation is reflexive, symmetric, transitive, antisymmetric. Is it an equivalence relation?

(c) Now consider the *taken before* relation which is defined to be the reflexive and transitive closure of the *prerequisite* relation. Draw the Hasse Diagram for the *taken before* relation.

(d) The new regulations place no limit on how many courses a student can take in one term. What is the minimum number of terms required to complete all the required courses? Why?

(e) Write down a maximum-size antichain in the partial order. What is the meaning of an antichain in terms of a student's schedule?

Problem 4. An *independent set* in a graph is a subset S of the vertices such that no edge connects two vertices in S. Consider the following algorithm on a graph with vertices $v_1, \ldots, v_n$:

initially no vertex is marked
for $i = 1$ to n
    if no neighbor of v_i is marked **then**
        mark v_i

(a) Prove that when this algorithm terminates, the set of marked vertices is an independent set.

(b) **(Optional)** Prove that the set of marked vertices is a *maximal* (under containment) independent set, in that adding any other vertex will make it not-independent.

Problem 5. We know that trees are bipartite because they contain no odd cycles. Give a direct proof that trees are bipartite by induction on the number of tree vertices.

Problem 6. Sums.

(a) Give tight (Θ) bounds for

$$\sum_{i=1}^{n} i^{1/3}.$$

(b) Express the following in closed form

$$\sum_{p=1}^{\infty} \sum_{q=p}^{\infty} x^p y^q.$$

(Note that the lower indices are not constants.)

Problem 7. Consider the following recurrence:

$$T(n) = 2T(\lfloor n/2 \rfloor) + n \log\log n, \qquad T(1) = 1$$

(a) Assume that n is a power of 2 so that the floor operation can be ignored. Give tight (Θ) bounds for $T(n)$. Show your derivation, but don't bother proving the result correct. (Hint: you might use Stirling's approximation to asymptotically bound $\log((\log n)!)$.

(b) Prove that $T(n)$ is an increasing function of n (that is, that $T(n+1) > T(n)$).

(c) From the previous two parts, deduce tight (Θ) bounds on $T(n)$ when n is an arbitrary natural number.

Problem 8. Rank the following functions by order of growth; that is find an arrangement $g_1 = \alpha(g_2) = \alpha(g_3) = \ldots = \alpha(g_6)$ and state whether α is o or Θ for each step.

- $n(\log_2 n)^5$
- n^3
- $n!$
- $(n!)^2$
- n^n
- $3 \cdot 8^{\log_2 n} + 4^{\log_2 n}$

Problem 9. Recurrences.

(a) Suppose that each pair of a genetically engineered species of rabbits left on an island produces two new pairs of rabbits at the age of one month and six new pairs of rabbits at the age of two months and every month afterwards. None of the rabbits ever die or leave the island. Find a recurrence relation for the number of *pairs* of rabbits on the island n months after one newborn pair is left on the island. Do not solve the recurrence relation.

(b) Solve the following inhomogeneous recurrence relation:

$$f(n) = 6f(n-1) - 9f(n-2) + 2^n, n \geq 2$$

The initial conditions are $f(0) = 5, f(1) = 26$.

Problem 10. Consider the following nuclear reaction. There is one particle of type 1 at time instant 1. Each particle of type i existing at time t undergoes fission to produce one particle of type $i+1$ and one of type $i+2$ at time instant $t+1$.

(a) Write and justify a recurrence for $f(i,t)$, the number of particles of type i at time t. (Make sure to include the base cases for your recurrence.)

(b) Prove by induction that $f(i,t) = \binom{t-1}{i-t}$. (You do not need to reprove identities that were proven in class.)

Problem 11. How many 6-digit decimal numbers are there that do not contain "123" or "456" as a subsequence? (for purposes of this problem numbers may have leading 0's)

Problem 12. Binomial Coefficients.

(a) Prove, by direct calculation, that the following binomial identity is true for all l, m, n such that $0 \leq l \leq m \leq n$:

$$\binom{n}{m}\binom{m}{l} = \binom{n}{l}\binom{n-l}{m-l}$$

(b) Now prove the same identity combinatorially. Let A be an n-element set. Explain how the left-hand and right-hand sides of the identity in part a) may be interpreted as the number of ways of choosing two subsets B and C of A, where $|B| = m$, $|C| = l$, and $C \subseteq B$.