Problem Set 8

Due: April 10, 1997.

Reading: Rosen 7.1-7.4 (skip isomorphism), 7.8, 8.1

Problem 1. Answer the following questions concerning the poset consisting of the elements {{a},{b},{d},{a,b},{a,d},{b,d},{c,d},{a,c,d},{b,c,d}} under $\subseteq$.

(a) Draw the Hasse diagram for this poset.

(b) Find the maximal elements

(c) Find the minimal elements

(d) Is there a greatest element?

(e) Is there a least element?

(f) Find all upper bounds of {{b},{d}}.

(g) Find the least upper bound of {{b},{d}} if it exists.

(h) Find all lower bounds of {{a,c,d},{b,c,d}}.

(i) Find the greatest lower bound of {{a,c,d},{b,c,d}} if it exists.

Problem 2. Each of the following sets of people needs to hold a meeting. Draw a conflict graph for these meetings, and schedule them into the smallest possible number of time slots without sending someone to two meetings at the same time.

$$\begin{aligned}
C_1 &= \{\text{Karger, Sussman, Abelson}\}\\
C_2 &= \{\text{Sussman, Ward, Minsky}\}\\
C_3 &= \{\text{Karger, Minsky, Abelson}\}\\
C_4 &= \{\text{Ward, Minsky, Abelson}\}\\
C_5 &= \{\text{Karger, Sussman}\}\\
C_6 &= \{\text{Sussman, Minsky, Abelson}\}
\end{aligned}$$

Problem 3. Give the adjacency matrix of the graph W_4. How many (simple or not) paths of length 2, 3 and 4 are there between the center vertex and each other vertex?

Problem 4. Prove that any tree with a degree k vertex has at least k leaves.

Problem 5. What is wrong with the following "theorem" and "proof?"

False theorem: If every vertex in a graph has degree at least one then the graph is connected.

False proof: We prove the result by induction on the number of vertices. Let $P(n)$ be "if every vertex in a graph with n vertices has degree at least one then the graph is connected."

Base case. $P(2)$ is true since the only graph satisfying the assumption is that with two vertices and one edge between them which is easily seen to be connected.

Inductive step. Assume $P(n)$ is true. Now, take any graph with n vertices with minimum degree at least one and add a new vertex to it. By inductive hypothesis the graph on n vertices is connected. Since the new vertex satisfies the assumption it must have degree at least one, so it must have an edge connecting it to one of the remaining n vertices. Thus the whole graph is connected. Hence $P(n+1)$ is true. Thus, $P(n)$ is true for all n.

The next problems are going to prove an interesting fact about graphs that that can be colored with two colors. These graphs are called *bipartite* because you can divide them into two groups (colors) such that every edge has one endpoint in each group. We prove:

Theorem 0.1 *A graph is bipartite if and only if it contains no cycles of odd length.*

It is enough to prove this theorem for a connected graph, because then we can apply it separately to each component of an arbitrary graph.

Problem 6. Prove that a graph contains an odd-length circuit if and only if it contains an odd length cycle (hint: consider the shortest odd length circuit).

Problem 7. Prove that if G is bipartite then it contains no circuits of odd length.

Problem 8. Suppose G is a connected graph with no odd cycles (or circuits). Fix a particular vertex u.

(a) Prove that for any vertex v, the length of every path from u to v is odd or the length of every path from u to v is even.

(b) Suppose I color a vertex accoring to the odd/evenness of its paths from u. Prove this is a legal coloring. Deduce that G is bipartite.

Optional Problem 9. Suppose we are in a connected maze (undirected graph) and explore it according to the following rules:

1. Never traverse an edge more than once in each direction
2. The first time we enter a room, mark our entering edge. Do not use the entering edge to exit a room unless it is the only remaining choice under the first rule.

Prove that we get stuck (can't apply either rule) only at our starting point, and that by then every edge has been traversed in both directions.