Practice Quiz 1 Solutions

Problem 1. *Prove by induction that*

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2.$$

Base case: n=1

$1^3 = 1^2 = 1$

Inductive Step: $H(n) \rightarrow H(n+1)$

$$\begin{aligned}
\left(\sum_{i=1}^{n+1} i\right)^2 &= \left(\sum_{i=1}^{n} i + (n+1)\right)^2 \\
&= \left(\sum_{i=1}^{n} i\right)^2 + 2(n+1)(\sum_{i=1}^{n} i) + (n+1)^2 \\
&= \sum_{i=1}^{n} i^3 + 2(n+1)(\sum_{i=1}^{n} i) + (n+1)^2 \\
&= \sum_{i=1}^{n} i^3 + 2(n+1)n(n+1)/2 + (n+1)^2 \\
&= \sum_{i=1}^{n} i^3 + (n+1)n(n+1) + (n+1)^2 \\
&= \sum_{i=1}^{n} i^3 + (n+1)^3 \\
&= \sum_{i=1}^{n+1} i^3
\end{aligned}$$

Problem 2. *An infinite sequence of natural numbers $a_1, a_2, a_3, \ldots$ is defined in such a way that $a_1 = 14, a_2 = 35$, and each later a_i, $i \geq 3$, is obtained using one of the following rules:*

Rule A *Choose any two preceding elements of the sequence and define a_i to be their sum.*

Rule B *Choose any number $k \geq 1$ of preceding elements of the sequence, and define a_i to be their product divided by 7^{k-1}.*

Use the well-ordering principle to prove that every element in the sequence is a multiple of 7.

We note that a_1, a_2 are both divisible by 7.

Let S be the set of numbers, a_i, which are not divisible by 7, where $i \geq 3$.

If S is non-empty, then by well-ordering, there is a least element, a_m. If a_m is formed by rule A, then $a_m = a_i + a_j$ where $i < k$ and $j < k$. Since a_m is the least element in S, a_i and a_j are divisible by 7, and so $a_i + a_j$ is divisible by 7, so $7 \mid a_m$, and we have a contradiction.

If a_m is formed by rule B, we have $a_m = a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_k}$ where all $a_{i_j} < a_m$. Since a_m is the least element of S, $7 \mid a_{i_j}$ for all $1 \leq j \leq k$, and so, k 7's can be factored out of a_m and $7 \mid (a_m/7^{k-1})$. Again, we have a contradiction.

So S must be empty, and all a_i are divisible by 7.

Problem 3. Number theory.

(a) *Using Euclid's algorithm, or otherwise, find an x such that*

$$43x \equiv 4 \pmod{105}.$$

x	y	q	r	Equation
105	43	2	19	$105 = 43 \cdot 2 + 19$
43	19	2	5	$43 = 19 \cdot 2 + 5$
19	5	3	4	$19 = 5 \cdot 3 + 4$
5	4	1	1	$5 = 4 \cdot 1 + 1$

Looking at the above equations and going in backwards order, we have

$$\begin{aligned} 1 &= 5 - 4 \\ &= 5 - (19 - 5 \cdot 3) \\ &= 5 \cdot 4 - 19 \\ &= (43 - 19 \cdot 2) \cdot 4 - 19 \\ &= 43 \cdot 4 - 19 \cdot 9 \\ &= 43 \cdot 4 - (105 - 43 \cdot 2) \cdot 9 \\ &= 43 \cdot 22 - 105 \cdot 9. \end{aligned}$$

and multiplying by 4

$$\begin{aligned} 1 &= 43 \cdot 22 - 105 \cdot 9 \\ 4 &= 43 \cdot 88 - 105 \cdot 36 \end{aligned}$$

So, $x = 88$.

(b) *Prove that if p and q are distinct primes, and $p \mid a$ and $q \mid a$, then $pq \mid a$.*
Find the prime factorization of $a = p_1p_2p_3 \dots p_n$. If $p \mid a$, then for some i, $p_i = p$. Similarly, for some j, $p_j = q$. Rearranging the factors (by commutativity), we have $a = (p_ip_j)p_1p_2p_3 \dots p_n$. So, $p_ip_j \mid a$ which means that $pq \mid a$.

Problem 4. Which of the following functions are injections? surjections? bijections? If a function is a bijection write down its inverse.

(a) $f : \mathbb{R} \to \mathbb{R}, f(x) = x$
injection, surjection, bijection, $f^{-1}(x) = x$

(b) $f : \mathbb{R}^+ \to \mathbb{R}^+, f(x) = \frac{1}{x}$
injection, surjection, bijection, $f^{-1}(x) = 1/x$

(c) $f : \mathbb{N} \to \mathbb{N}, f(x) = x^2$
injection

(d) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$ injection, surjection, bijection, $f^{-1}(x) = x^{1/3}$

Problem 5. Let $f : U \to V$ be an arbitrary function and A, B be arbitrary subsets of U. Prove or disprove the following claims (recall that $f(A) = \{f(a) \mid a \in A\}$.
a) $f(A \cap B) = f(A) \cap f(B)$

Disprove:
Take $f = \{(2, 4), (5, 4)\}$.
Let $A = \{2\}$, $B = \{5\}$ $f(A) = \{4\}$, $f(B) = \{4\}$,
$A \cap B = \emptyset$
$f(\emptyset) = \emptyset$
$f(A) \cap f(B) = \{4\}$
so $f(A \cap B) \neq f(A) \cap f(B)$

b) $f(A \cup B) = f(A) \cup f(B)$

Prove true:
1. Show $x \in f(A \cup B) \to x \in (f(A) \cup f(B))$
Assume $x \in f(A \cup B)$, i.e., $f(y) = x$ for some $y \in A \cup B$. Either $y \in A$ and $x = f(y) \in f(A)$ or $y \in B$ and $x = f(y) \in f(B)$. In both cases $y \in f(A) \cup f(B)$. 2. Show $x \in f(A) \cup f(B) \to x \in f(A \cup B)$
Let $x \in f(A) \cup f(B)$. Either $x = f(y)$ for some $y \in A$ or $x = f(y)$ for some $y \in B$. In both cases $x = f(y)$ for some $y \in A \cup B$ and therefore $x \in f(A \cup B)$.

Problem 6. Let $(a, b) \sim (c, d)$ if $a + d = b + c$.

(a) Prove that $\sim$ is an equivalence relation on $\mathbb{Z} \times \mathbb{Z}$.

And equivalence class is reflexive, symmetric, and transitive.

$(a, b) \sim (a, b)$ for all (a, b) since $a + b = a + b$, so $\sim$ is reflexive.

If $(a, b) \sim (c, d)$ then $(c, d) \sim (a, b)$ because if $a+d = b+c$ then also $c+b = d+a$, so $\sim$ is symmetric.

If $(a, b) \sim (c, d) \wedge (c, d) \sim (e, f)$ then $(a, b) \sim (e, f)$ because $(a+d = b+c) \wedge (c+f = d + e)$ implies $a + d + c + f = b + c + d + e$ and therefore $a + f = e + b$.

(b) Partition the set $\{0, 1, 2\} \times \{0, 1, 2\}$ into the distinct equivalence classes under $\sim$.

The equivalence classes of the above set are $\{(0, 0), (1, 1), (2, 2)\}$, $\{(0, 1), (1, 2)\}$, $\{(0, 2)\}$, $\{(1, 0), (2, 1)\}$ and $\{(2, 0)\}$.

(c) How many distinct equivalence classes under $\sim$ are there on the set $\{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\}$? You do not need to prove your answer.

There are $2n+1$ equivalence classes corresponding to representatives $(n, 0), (n-1, 0), \ldots, (1, 0), (0, 0), (1, 0), \ldots, (n, 0)$.

Problem 7. Consider the following algorithm for performing subtraction by repeated decrements. It takes two natural numbers a and b.

Sub(a, b)

$x \leftarrow a$
$y \leftarrow b$
while $(y > 0)$
  $x \leftarrow x - 1$
  $y \leftarrow y - 1$
return x.

(a) Prove that this algorithm terminates

Since, $b \geq 0$ and $y \geq 0$, and either the while loop is never entered because $y = 0$, or y continually decrements to 0.

(b) Prove that this algorithm returns $a - b$.

Proof by induction on the number of iterations, i, through the while loop. $H(i) =$ after the ith iteration, $x = a - i$ and $y = b - i$.

Base Case: H(1)

$x = a - 1$, and $y = b - 1$.

Inductive Step:

Assume $H(m)$, show $H(m + 1)$. $x_{m+1} = x_m - 1$ and $y_{m+1} = y_m - 1$

by the inductive hypothesis, $x_{m+1} = a - i - 1 = a - (i + 1)$ and $y_{m+1} = b - i - 1 = b - (i + 1)$

Problem 8. You are given the following operations on a string:

FIRST(w) – returns the first character of a string or ϵ if $w = \epsilon$
REST(w) – returns the string w minus w's first character. if $w = \epsilon$, then returns ϵ
EMPTY(w) – returns TRUE if w is ϵ, FALSE otehrwise.

The alphabet in this case is 0,1. $a \cdot w$ is the character or string a, concatenated with w.

```
Separate(w)
    if (EMPTY(w)) return (ε);
    else if (FIRST(w)='0')
        return ('0' · separate(REST(w)));
    else
        return (separate(REST(w))·'1');
```

(a) Prove that this algorithm terminates.

Inductive hypothesis: $H(i)$: Separate(w) where w is i characters long, terminates.

Base case:

H(0): EMPTY(w) is true terminates.

Inductive step:

Assume $H(n)$, show termination for $H(n+1)$.

if (FIRST='0'), then Separate(w) terminates if the call to Separate(REST(w)) terminates. REST(w) only has n characters, so it terminates by the inductive hypothesis. The argument for FIRST(w)='1' is similar, so Separate(w) terminates.

(b) Prove that if the input to the above algorithm is a string with a 0's and b 1's, then the output is the string consisting of a 0's followed by b 1's

Inductive Hypothesis: H(n): if w is a string of length n, with i 0's and j 1's, $i + j = n$, then Separate(w) returns a string with i 0's followed by j 1's.

If (FIRST(w) ='0') then there are $i - 1)$0's and j 1's in REST(w), and Separate(REST(w)) returns $i - 1$ 0's followed by j 1's by the inductive hypothesis. So Separate(w) returns $1 + (i - 1) = i$ 0's and j 1's.

Similarly, if (FIRST(w) = '1'), then there are i 0's and $j-1$ 1's in REST(w), and Separate(REST(w)) returns i 0's followed by $j-1$ 1's by the inductive hypothesis. So Separate(w) returns i 0's and $j - 1 + 1 = j$ 1's.

Problem 9. The Massachusetts Transit Authority has just built a new one-way bridge that will collapse if more than 60 cars are on it at the same time. To prevent this from happening, they've put a 25 cent toll both A at the entrance and a 25 cent toll booth B at the end. A car deposits 25 cents at A go onto the bridge and 25 cents at B to get off. At A, they only let cars on when the difference between the amount of money at A and at B is less than 15 dollars.

(a) Describe a state machine that models the bridge, and list its transitions.

Variables:
A: Amount of money at bridge A
B: Amount of money at bridge B
cars: number of cars on the bridge

Initial conditions:
$A = B = cars = 0$

Transitions:
BridgeA(car)
Precondition:
$A - B < 15$
there's a car at the A's booth
effects:
$A = A + 0.25$
$cars = cars + 1$

BridgeB(car)
$A - B < 15$
there's a car at the B's booth
effects:
$B = B + 0.25$
$cars = cars - 1$

(b) Prove that there will never be too many cars on the bridge.

Invariant:
$4(A - B) = cars \leq 60$
Initial conditions: $0 = 0 \leq 60$
After Transition A:
$A_{new} = A + 0.25$ and $cars_{new} = cars + 1$ so $4(A_{new} - B) = 4((A + 0.25)) - B) = 4(A - B) + 1 = cars + 1 = cars_{new}$. Also, $4(A - B) = cars < 60 \rightarrow cars + 1 = cars_{new} \leq 60$, (because the precondition has $A - B < 15$).
After Transition B:
$B_{new} = B + 0.25$ $cars_{new} = cars + 1$ so $4(A - B_{new}) = 4(A - (B - .025)) = 4(A - B) - 1 = cars - 1 = cars_{new}$. Also, since $cars \leq 60$ to start, it will still be $cars \leq 60$.

So the invariant is satisfied and the number of cars is never greater than 60.