Practice Quiz 1

Problem 1. Prove by induction that

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2.$$

Problem 2. An infinite sequence of natural numbers $a_1, a_2, a_3, \ldots$ is defined in such a way that $a_1 = 14, a_2 = 35$, and each later a_i, $i \geq 3$, is obtained using one of the following rules:

Rule A Choose any two preceding elements of the sequence and define a_i to be their sum.

Rule B Choose any number $k \geq 1$ of preceding elements of the sequence, and define a_i to be their product divided by 7^{k-1}.

Use the well-ordering principle to prove that every element in the sequence is a multiple of 7.

Problem 3. Number theory.

(a) Using Euclid's algorithm, or otherwise, find an x such that

$$43x \equiv 4 \pmod{105}.$$

(b) Prove that if p and q are distinct primes, and $p \mid a$ and $q \mid a$, then $pq \mid a$.

Problem 4. Which of the following functions are injections? surjections? bijections? If a function is a bijection write down its inverse.

(a) $f : \mathbb{R} \to \mathbb{R}, f(x) = x$

(b) $f : \mathbb{R}^+ \to \mathbb{R}^+, f(x) = \frac{1}{x}$

(c) $f : \mathbb{N} \to \mathbb{N}, f(x) = x^2$

(d) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$

Problem 5. Let $f : U \to V$ be an arbitrary function and A, B be arbitrary subsets of U. Prove or disprove the following claims (recall that $f(A) = \{f(a) \mid a \in A\}$.

a) $f(A \cap B) = f(A) \cap f(B)$

b) $f(A \cup B) = f(A) \cup f(B)$

Problem 6. Let $(a, b) \sim (c, d)$ if $a + d = b + c$.

(a) Prove that $\sim$ is an equivalence relation on $\mathbb{Z} \times \mathbb{Z}$.

(b) Partition the set $\{0, 1, 2\} \times \{0, 1, 2\}$ into the distinct equivalence classes under $\sim$.

(c) How many distinct equivalence classes under $\sim$ are there on the set $\{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\}$? You do not need to prove your answer.

Problem 7. Consider the following algorithm for performing subtraction by repeated decrements. It takes two natural numbers a and b.

Sub(a, b)

$x \leftarrow a$
$y \leftarrow b$
while $(y > 0)$
    $x \leftarrow x - 1$
    $y \leftarrow y - 1$
return x.

(a) Prove that this algorithm terminates

(b) Prove that this algorithm returns $a - b$.

Problem 8. You are given the following operations on a string:

FIRST(w) – returns the first character of a string or ϵ if $w = \epsilon$
REST(w) – returns the string w minus w's first character. if $w = \epsilon$, then returns ϵ
EMPTY(w) – returns TRUE if w is ϵ, FALSE otehrwise.

The alphabet in this case is 0,1. $a \cdot w$ is the character or string a, concatenated with w.

```
Separate(w)
    if (EMPTY(w)) return (ε);
    else if (FIRST(w)='0')
        return ('0' · separate(REST(w)));
    else
        return (separate(REST(w))·'1');
```

(a) Prove that this algorithm terminates.

(b) Prove that if the input to the above algorithm is a string with a 0's and b 1's, then the output is the string consisting of a 0's followed by b 1's

Problem 9. The Massachusetts Transit Authority has just built a new one-way bridge that will collapse if more than 60 cars are on it at the same time. To prevent this from happening, they've put a 25 cent toll both A at the entrance and a 25 cent toll booth B at the end. A car deposits 25 cents at A go onto the bridge and 25 cents at B to get off. At A, they only let cars on when the difference between the amount of money at A and at B is less than 15 dollars.

(a) Describe a state machine that models the bridge, and list its transitions.

(b) Prove that there will never be too many cars on the bridge.