Problem Set 7

Due: March 31, 1997.

Reading: Rosen 4.2. Lecture notes 10,11,12

Problem 1. Show that for all odd natural numbers m there exists an $n \geq 1$ such that $m \mid 2^n - 1$.

Problem 2. Jelly beans of 8 different colors are in 6 jars. There are 20 jelly beans of each color. Use the pigeonhole principle to prove that there is some jar that contains at least two beans each from two two different colors of jelly beans.

Problem 3. Let F be the set of all functions $f : \mathbb{R} \to \{0, 1\}$. Show that there is a bijection from $2^{\mathbb{R}}$ to F. This is another reason for denoting power-set by $2^{\mathbb{R}}$. More generally, A^B is used to denote the set of all functions from B to A (in particular, for set theorists who only care about cardinalities, 2 denotes a two-element set).

Problem 4. Let P be a relation on a finite set S that makes S into a poset. Define a function f on S by $f(x) =$ the number of elements y of S for which $(x, y) \in P$. Define R to be the relation $\{(x, y) : x, y \in S \text{ and } f(x) \geq f(y)\}$.

(a) Prove that R also makes S into a poset.

(b) Prove that $P \subseteq R$.

Problem 5. Minimal elements of posets.

(a) Let (S, R) be a poset. Show that (S, R^{-1}) is also a poset, where R^{-1} is the inverse of R.

(b) In class we used piegonhole to show that any finite poset has a minimum element. Give a pigeonhole-free proof using induction on the size of the poset (that is, number of elements in the underlying set).

(c) Combining the previous two parts, prove that every poset has a *maximal* element that is not less than or equal to any other.

Problem 6. A real number is *algebraic* if it is the root of a polynomial with rational number coefficients. Algebraic numbers form an important part in the study of (what else) modern algebra. A number that is not algebraic is called *transcendental.*

(a) Let $S_1, S_2, S_3, \ldots$ be a collection of countably many sets. Suppose each S_i is countable. Prove $\cup S_i$ is countable.

(b) Prove that for any particular d, the set of degree d polynomials with rational coefficients is countable.

(c) Prove that the set of all roots of degree d polynomials with rational coefficients is countable (you may use the fact that a degree d polynomial has at most d roots).

(d) Prove that the set of algebraic numbers (which is just the union of the sets in the previous part) is countable.

(e) Prove that there is a trancendental number.

This proof was originally given by Cantor (who invented diagonalization). A lot of mathematicians of the time rejected it because it didn't exhibit any particular trancendental number (it is *non-constructive*. Since then, people have demonstrated that many important numbers (e.g. e and π) are trancendental.

Optional Problem 7. I stated in class that to prove countability it suffices to show that there is a surjection from $\mathbb{N}$ to a set. Prove this statement. That is, Suppose that S is not finite and we have a surjection $f : \mathbb{N} \to S$. Prove that there is a bijection from $\mathbb{N}$ to S.

This proves that there is no countably infinite set "smaller" than $\mathbb{N}$. Since we know that (by definition) there is no set that is "bigger", we have shown that all countably infinite sets are the same size.