Problem Set 6

Due: March 18, 1997.

This problem set has **only 3** required problems, numbers 1–3. **All** others are **optional.** If you do them, they will be corrected (which will give you useful feedback for next time) but the grade will not count. I suggest you use them to study for the quiz, since they cover material that will be quizzed.

Reading: Rosen, 1.5–1.6, 6.1–6.5.

Problem 1. There is a ferry boat which leaves from Boston Harbor. In order to board the boat, a person must go through several steps:

- Go to the ticket counter
- Purchase a ticket at the ticket counter
- Go to the boat
- Give the purser to the conductor and get on

(Tickets purchased for a particular trip are not good for any other trips). Give a state machine that models this scenario, in which each of the above steps corresponds to one (separate) transition.

Suppose that initially there are only 60 tickets at the ticket counter. Add this fact to your state machine, and prove that there are never more than 60 passengers on the boat.

Problem 2. Consider an array/permutation $A[]$. Show that it is impossible to implement a two-element swap (eg, exchange elements 1 and 2 without changing any of the other elements' positions) via any sequence of 3-element cyclic shifts (eg, $1 \to 2 \to 3 \to 1$)). Hint: an inversion in a permutation is a pair of elements i and j with $i < j$ but $A[i] > A[j]$. Study the number of inversions in the permutation as you perform swaps and shifts.

Problem 3. Let R_1 and R_2 be two equivalence relations on a set S. Show that $R_1 \cap R_2$ is an equivalence relation on S.

Optional Problem 4 Let A, B, and C be sets. Prove that $A - (B \cap C) = (A - B) \cup (A - C)$.

Optional Problem 5 Prove that for any relations Q, R and S

$$Q \circ (R^c \circ S)^c \subseteq ((Q \circ R)^c \circ S)^c.$$

Optional Problem 6 Let X and Y be sets, and $f : X \to Y$ be a function. Recall that for a set X, $f(X) = \{f(x) \mid x \in X\}$. Prove or disprove the following.

(a) For any subsets A and B of X, $f(A) \cup f(B) = f(A \cup B)$

(b) For any subsets A and B of Y, $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B)$

Optional Problem 7 Determine which of the relation following relations R on all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ iff:

(a) a is taller than b

(b) a and b were born on the same day of the week

(c) a and b were born less than 24 hours apart

(d) a has a different first name from b

(e) a and b have a common grandparent

Optional Problem 8 Define a relation R on $\mathbb{N}$ by, $(a, b) \in R$ if $\exists c \in \mathbb{N}$ such that $a^2 = b^2 + c^2$.

Decide whether R is reflexive, symmetric, antisymmetric, or transitive.

Optional Problem 9 Show that the transitive closure of the symmetric closure of the reflexive closure of a relation R is the smallest equivalence relation that contains R.

Optional Problem 10 In the next problems, we are going to define the integers as a equivalence classes of pairs of natural numbers. For these problems, use only natural numbers and the properties of addition. Don't use negation or subtraction. You may use the fact that addition is associative, distributive, and commutative. You will also find it useful to use the following cancellation lemma: if $a+b = a+c$, then $b = c$ (this can be proved by induction from the Peano-axiom definition of plus, but you can take it as an axiom).

We define a relation $\equiv$ on $\mathbb{N} \times \mathbb{N}$ by $(a, b) \equiv (c, d)$ iff $a + d = b + c$. Prove that $\equiv$ is an equivalence relation.

Optional Problem 11 Let $\mathbb{Z}$ be the set of equivalence classes of $\equiv$ (from the previous problem). We define a relation P from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$ by the following rule. $(X, Y) \sim_P Z$ iff $\exists a, b, c, d \in \mathbb{N}$ for which $(a, b) \in X$, $(c, d) \in Y$, and $(a + c, b + d) \in Z$. Note that $\sim_P$ is a binary relation, but its first argument is an ordered pair.

(a) Prove that $\sim_P$ is actually a function from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$.

(b) Since P is a function, we will let $X + Y$ denote the (unique) Z for which $(X, Y) \sim_P Z$. We now define subtraction via a relation M as follows. $(X, Y) \sim_M Z$ if $X = Y + Z$. Prove that M is a function. Let $X - Y$ denote the Z for which $M((X, Y), Z)$.

Optional Problem 12 We now prove some properties of our plus and minus functions.

(a) Prove that there is an element Θ in $\mathbb{Z}$ such that $\forall X \in \mathbb{Z} \quad X + W = W + X = X$.

(b) Prove that for every element X in $\mathbb{Z}$, there is an element Y in $\mathbb{Z}$ such that $X + Y = \Theta$.

(c) Prove that $X - X = \Theta$.

(d) Give an injective function f from $\mathbb{N}$ to $\mathbb{Z}$ such that for any $x, y \in \mathbb{N}$, $f(x+y) = f(x)+f(y)$ (note that the right-hand side plus is operating on $\mathbb{Z}$). This function puts of "copy" of the natural numbers into the integers we've defined.