Problem Set 3 Solutions

Problem 1. Show that the Fibonacci number f_n satisfy $f_n \leq 2^n$ using strong induction.

BASE CASES ($n = 0, 1$): If $n = 0$ then $f_0 = 0$ which is less then or equal to $2^0 = 1$. If $n = 1$ then $f_1 = 1 \leq 2^1 = 2$.

INDUCTIVE HYPOTHESIS ($n \geq 1$): Assume that $f_m \leq 2^m$ for all $m \leq n$.

INDUCTIVE STEP: We use the inductive hypothesis to prove that $f_{n+1} \leq 2^{n+1}$. Using the recursive definition of fibonacci number we get $f_{n+1} = f_n + f_{n-1}$ which by inductive hypothesis is less then or equal to $2^n + 2^{n-1} < 2^n + 2^n = 2^{n+1}$.

First of all notice that if we can make n cents using dimes and quarters then $n = 10a + 25b = 5(2a + 5b)$ for some a and b, and therefore n must be a multiple of 5. We can make 10 cents using one dime, and we cannot make neither 5 nor 15 cents. We claim that we can make any other quantities n such that $n \geq 20$ and $5|n$. In other words, we can make $20 + 5k$ cents for all $k \geq 0$. We will prove this statement first by strong induction and then by well-ordering.

(a) We prove that we can make $20 + 5k$ cents using dimes and quarters by strong induction on k.

BASE CASES ($k = 0, 1$) We can make 20 and 25 cents using two dimes and using one quarter respectively.

INDUCTIVE STEP ($k \geq 2$): Assume we can make $20 + 5h$ cents for all $h < k$. In particular we can make $20 + 5(k - 2)$ cents using dimes and quarters. Using one more dime, we can also make $20 + 5(k - 2) + 10 = 20 + 5k$ cents, completing the induction proof.

(b) We prove that we can make $20 + 5k$ cents using dimes and quarters by well-ordering. Let S be the set of all k such that we cannot make $20 + 5k$ cents using dimes and quarters and assume for contradiction that S is not empty. By well-ordering, S has a minimum element m. m cannot be 0 or 1 because we can make 20 and 25 cents. Therefore, $m \geq 2$ and $m - 2 \geq 0$. We claim that $m - 2$ is in S. To prove this, notice that if we could make $20 + 5(m - 2) = 20 + 5m - 10$ cents using dimes and quarters, then we could make also $20 + 5m$ cents using one more dime. But, this is impossible because $m \in S$. So, $m - 2 \in S$ and m is not the minimum.

Problem 2. Call a natural number *blue* if it can be written as a sum of one or more (not necessarily distinct) odd primes.

(a) Identify all blue numbers.

Blue numbers are 3 and all numbers greater then or equal to 5.

(b) Prove you answer to the previous part.

By inspection we notice that 3 is blue, and $0, 1, 2$ and 4 are not blue. It remains to be proven that all numbers $n \geq 5$ are blue. The proof is by induction on n.

Base Case ($n = 5, 6, 7$): $5 = 5$, $6 = 3 + 3$ and $7 = 7$.

Inductive Step ($n \geq 8$): Assume that all numbers between 5 and n (excluded) are blue. We want to prove that n is blue. Notice that $n - 3$ is blue by hypothesis. and therefore it can be written as a sum of odd primes $p_1 + p_2 + \cdots + p_k$. It follows that $n = (n - 3) + 3 = p_1 + p_2 + \cdots + p_k + 3$ is also blue.

Problem 3. Call a number *perfect* if it is the sum of all its divisors other than itself.

(a) Prove that 6 and 28 are perfect.

The number 6 is divisible by $1, 2, 3, 6$ and $1+2+3 = 6$. So, 6 is perfect. Similarly 28 is divisible by $1, 2, 4, 7, 14, 28$ and $1 + 2 + 4 + 7 + 14 = 28$. So, 28 is perfect.

(b) Prove that $n = 2^{p-1}(2^p - 1)$ is perfect when $2^p - 1$ is prime.

First of all notice that 2 and $2^p - 1$ are two distinct primes. Therefore the set of divisors of $n = 2^{p-1}(2^p - 1)$ is

$$\{2^i(2^p - 1)^j \mid 0 \leq i \leq p - 1, 0 \leq j \leq 1\}.$$

If we add up all these numbers we get:

$$\left(\sum_{i=0}^{p-1} 2^i\right) \cdot (1 + (2^p - 1)) = (2^p - 1) \cdot 2^p = 2n.$$

Notice that we added up all divisors of n (including itself). So, the sum of all divisors of n other then itself is $2n - n = n$, proving that n is perfect.

Problem 4. Read the recursive definition of strings over an alphabet Σ in Rosen 3.3, example 10. Define the *reversal* of a string w, written w^R, as follows:

- $\lambda^R = \lambda$
- if $w = vx$ where $v \in \Sigma^*$ and $x \in \Sigma$, then $w^R = xv^R$

Prove that $(vw)^R = w^R v^R$

We prove by structural induction on w that for all w and v, $(vw)^R = w^R v^R$.

Base Case: $w = \lambda$ and

$$(v\lambda)^R = v^R = \lambda v^R = \lambda^R v^R.$$

INDUCTIVE STEP: We have $w = zx$ for some $x \in \Sigma$. Using the inductive hypothesis $(vz)^R = z^R v^R$ and the definition of R we get

$$\begin{aligned} (vw)^R &= (vzx)^R \\ &= x(vz)^R \\ &= xz^R v^R \\ &= (zx)^R v^R \\ &= w^R v^R. \end{aligned}$$

Problem 5. Recall that we defined binary trees in class as either

- A *node* with no children.
- A node with a left child that is a binary tree, and no right child.
- A node with a right child that is a binary tree, and no left child.
- A node with left and right binary-tree children.

Consider the following recursive definition of the *height* of a binary tree:

- The height of a node with no children is 0
- The height of a tree on a node with children is one more than the maximum of its children's heights.

Prove that a tree of height h contains at most $2^{h+1} - 1$ nodes.

The proof is by structural induction on the definition of binary tree.

BASE CASE: A tree consisting of one node with no children has height $h = 0$. So $1 \leq 2^{h+1} - 1 = 1$.

INDUCTIVE STEPS: We have three cases corresponding to the three cases of the definition:

- If the tree T is a node with a left child but no right child, let L be the left child binary tree and let h be its height. By inductive hypothesis, the tree L has at most $2^{h+1} - 1$ nodes. The height of tree T is $h + 1$, and the number of nodes in T is at most $1 + 2^{h+1} - 1 = 2^{h+1} < 2^{h+2} - 1$.
- If the tree T is a node with a right child but no left child, it is analogous to the previous case.
- If the tree T is a node with both a left and right child L and R, then let h_1 and h_2 be the heights of the trees L and R respectively. Let also h be the maximum of h_1 and h_2. By inductive hypothesis the trees L and R have at most $2^{h_1+1} - 1 \leq 2^{h+1} - 1$ and $2^{h_2+1} - 1 \leq 2^{h+1} - 1$ nodes. So, the number of nodes in T is at most $1 + (2^{h+1} - 1) + (2^{h+1} - 1) = 2^{h+2} - 1 = 2^{h'+1} - 1$ nodes, where $h' = h + 1$ is the height of T.

Problem 6. Find the flaws in the following "proofs."

(a) CLAIM: $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.

BASIS STEP: $a^0 = 1$ is true by the definition of a^0.

INDUCTIVE STEP: Assume that $a^k = 1$ for all nonnegative integers k with $k \leq n$. Then note that $a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1$.

The flaw is that we should prove both $n = 0$ and $n = 1$ as bases cases because the inductive step uses the previous two values of n. Clearly $a^1 = 1$ is not true, unless $a = 1$, and therefore the proof is wrong.

(b) CLAIM: $\sqrt{2}$ is rational.

PROOF: Well-ordering. Let S be the set of all rational numbers greater than or equal to $\sqrt{2}$. This set contains a least element r by well-ordering. If this element is not $\sqrt{2}$, then there is another rational element between $\sqrt{2}$ and r (by denseness of the rationals), implying r is not the smallest, a contradiction. Thus we must have $r = \sqrt{2}$.

The flaw is that the well-ordering principle does not hold for the rationals, so we cannot use it to conclude that the above set has a minimum.

(c) CLAIM: every number can be described in a phrase of less than eighty characters.

PROOF: Well-ordering. Let S be the set of numbers that cannot be described in a phrase of less than eighty characters. If it is nonempty, it contains a smallest element. This is "The smallest number that cannot be described in less than eighty chracters." This is a description of that number in less than eighty characters, a contradiction. Since the assumption that S was nonempty yields a contradiction, S must be empty and the claim is proved.

The flaw in the above proof is that the set of numbers that can be described in less than eighty characters, is not a well-defined set.

Problem 7. **(Optional)** How many zeroes are there at the end of 100!? Prove it.

The number of zeros at the end of 100! is the highest n such that $10^n | 100!$, or equivalently $2^n | 100!$ and $5^n | 100!$.

Let's first find the highest n such that $5^n | 100!$. One every five numbers in $1 \cdot 2 \cdots 99 \cdot 100$ is a multiple of five. So, there are $100/5 = 20$ multiple of 5 between 1 and 100 (included). Four of these numbers are in fact multiple of $25 = 5^2$ (25,50,75 and 100). So, we must account for four more 5 factors. In conclusion the total number of 5 in the prime factorization of 100! is $20 + 4 = 24$ and $n = 24$ is the highest n such that $5^n | 100!$.

We note that $2^{24} | 100!$ because 100! is the product of 100 numbers half of which are even.

Therefore $n = 24$ is the highest n such that $10^n | 100!$.