Problem Set 4

Due: March 4, 1997.

Reading: Sections 2.3–2.5, and 3.4-3.5 of *Discrete Mathematics and its Applications* by Kenneth H. Rosen. Warning: we take a different approach than 3.5; it should be read for intuition rather than method.

Problem 1. Let n be a positive integer with no factors (other than 1) that are perfect squares. Let a be the sum of the factors of n. Let b be the sum of the squares of the factors of n. Let c be the sum of the factors of n^3. Prove using well-ordering that $a \cdot b = c$.

Problem 2. Fill in the missing proof from class, that if $p \mid a_1 a_2 \cdots a_k$, then $(\exists i)(p \mid a_i)$.

Problem 3. Show that if a, b, and m are positive integers such that $m \geq 2$ and $m \mid (a-b)$, then $\gcd(a, m) = \gcd(b, m)$.

Problem 4. Let a and b be positive integers. The least common multiple of a and b is defined to be the smallest positive integer they both divide. Prove that this quantity always exists. Prove that $a \cdot b = \gcd(a, b) \cdot \text{lcm}(a, b)$.

Problem 5. We are given two buckets, A and B, a water hose, a receptacle, and a drain. The buckets and receptacle are initially empty. The buckets are labelled with their capacities, which are the positive integers a and b, respectively. The receptacle can be used to store an unlimited amount of water, but has no measurement markings. Excess water can be dumped into the drain. Consider the following "Die Hard" operations:

- fill bucket A or bucket B from the hose
- pour from the receptacle to either A or B until it is full or the receptacle is empty (whichever happens first)
- empty bucket A or bucket B to the drain or the receptacle
- pour from A to B until either A is empty or B is full (whichever happens first)
- pour from B to A until either B is empty or A is full (whichever happens first)

Let $c = \gcd(a, b)$. Let g be a positive integer. Prove that we can put g gallons into the receptacle using the above operations if, and only if, $c \mid g$.

Problem 6. Use Euclid's algorithm to compute $c = \gcd(171, 63)$ and compute integers x and y for which $171x + 63y = c$ (show your work). Using the model of the "Die Hard" problem, give an explicit scheme for putting exactly 18 ounces of water into the receptacle (the final holder) using a 171 ounce and a 63 ounce bucket.

Problem 7. Prove the correctness of the following algorithm for calculating m^n. Assume that initially, for any call to Power(m, n), $n \geq 1$.

procedure Power(m, n)

if $(n = 1)$

  return(m)

else

  return(Power$(m, \lfloor n/2 \rfloor)*$ Power $(m, n - \lfloor n/2 \rfloor)$)

(note that $\lfloor x \rfloor$ truncates to an integer, so $\lfloor 5/2 \rfloor = 2$. See Rosen p. 68)

Problem 8. Prove that the following algorithm for computing the Fibonacci numbers is correct.

procedure Fib(n)

if $n = 0$ then $y := 0$

else

  $x := 0$

  $y := 1$

  for $i := 1$ to $n - 1$

    $z := x + y$

    $x := y$

    $y := z$

  end for

return(y)

(y is the nth Fibonacci number).

Problem 9. **(Optional).** Consider the "Die Hard" problems. Can you solve problems 5 and 6 if you are forbidden to ever remove water from the receptacle?