Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professor David R. Karger

Handout 6
February 20, 1996

Problem Set 3

Due: Start of Class, Tuesday, February 25.

Reading: Rosen, Sections 2.3 and 3.3

Problem 1. Show that the Fibonnacci number f_n satisfy $f_n \leq 2^n$ using strong induction.

Problem 2. What quantities of money can be formed from just dimes and quarters?

(a) Prove by strong induction that these and no other quantities can be formed.

(b) Prove the same thing by well ordering

Problem 3. Call a natural number *blue* if it can be written as a sum of one or more (not necessarily distinct) odd primes.

(a) Identify all blue numbers

(b) Prove you answer to the previous part.

Problem 4. Call a number *perfect* if it is the sum of all its divisors other than itself.

(a) Prove that 6 and 28 are perfect.

(b) Prove that $2^{p-1}(2^p - 1)$ is perfect when $2^p - 1$ is prime.

Problem 5. Read the recursive definition of strings over an alphabet Σ in Rosen 3.3, example 10. Define the *reversal* of a string w, written w^R, as follows:

- $\lambda^R = \lambda$
- if $w = vx$ where $v \in \Sigma^*$ and $x \in \Sigma$, then $w^R\ xv^R$

Prove that $(vw)^R = w^R v^R$

Problem 6. Recall that we defined binary trees in class as either

- A *node* with no children.
- A node with a left child that is a binary tree, and no right child.
- A node with a right child that is a binary tree, and no left child.
- A node with left and right binary-tree children.

Consider the following recursive definition of the *height* of a binary tree:

- The height of a node with no children is 0
- The height of a tree on a node with children is one more than the maximum of its children's heights.

Prove that a tree of height h contains at most $2^{h+1} - 1$ nodes.

Problem 7. Find the flaws in the following "proofs."

(a) CLAIM: $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.

BASIS STEP: $a^0 = 1$ is true by the definition of a^0.

INDUCTIVE STEP: Assume that $a^k = 1$ for all nonnegative integers k with $k \leq n$. Then note that $a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1$.

(b) CLAIM: $\sqrt{2}$ is rational.

PROOF: Well ordering. Let S be the set of all rational numbers greater than or equal to $\sqrt{2}$. This set contains a least element r by well ordering. If this element is not $\sqrt{2}$, then there is another rational element between $\sqrt{2}$ and r (by denseness of the rationals), implying r is not the smallest, a contradiction. Thus we must have $r = \sqrt{2}$.

(c) CLAIM: every number can be described in a phrase of less than eighty characters.

PROOF: Well ordering. Let S be the set of numbers that cannot be described in a phrase of less than eighty characters. If it is nonempty, it contains a smallest element. This is "The smallest number that cannot be described in less than eighty chracters." This is a description of that number in less than eighty characters, a contradiction. Since the assumption that S was nonempty yields a contradiction, S must be empty and the claim is proved.

Problem 8. **(Optional)** How many zeroes are there at then end of 100!? Prove it.