Problem Set 1

Due: Start of Class, Tuesday, February 11.

Problem 1. Let p, q, and r be the following propositions:

p : You are a good water-polo player.
q : You are a good swimmer.
r : You are in the water-polo team.

Write the following propositions using p, q, and r and logical connectives (e.g. $\vee$, $\wedge$, $\neg$, $\Rightarrow$).

(a) To be in the water-polo team, it is sufficient for you to be a good water-polo player.

(b) Being a good swimmer and a good water-polo player is necessary for getting into the water-polo team.

(c) You will get in the water polo team if and only if you are a good swimmer or you are a good water-polo player.

(d) You are neither a good swimmer nor a good water-polo player, yet you are in the water-polo team.

Problem 2. Find a compound proposition involving the propositions p, q, and r that is false when exactly two of p, q, and r are false. The compound proposition is true otherwise.

Problem 3. Let $T(x, y)$ be the statement "x has proven y." Express each of the following sentences in terms of $T(x, y)$, quantifiers and logical connectives, where the universe of the discourse for x is the set of all mathematicians and for y is the set of all theorems.

a) There is a mathematician who has never proven a theorem.

b) Every mathematician has proven a theorem.

c) Every theorem has been proven by a mathematician.

d) There is a mathematician who has proven the *Dominated Convergence Theorem* and the *Bounded Convergence Theorem.*

e) At least two mathematicians have proven the *Hahn-Banach Theorem* (Don't introduce numbers. You will have to add one more standard symbol for this one).

Problem 4. (Due to Lewis Carrol). Define predicates and state the following arguments as predicate logic formulas.

(a) No frogs are poetical; some ducks are unpoetical. Thus, some ducks are not frogs.

(b) Gold is heavy; nothing but gold will silence him. Thus, nothing light will silence him.

(c) All lions are fierce; some lions do not drink coffee. Thus, some creatures that drink coffee are not fierce.

(d) Some pillows are soft, no pokers are soft. Thus, some pokers are not pillows.

Problem 5. Prove the correct arguments from the previous problem and show the incorrect ones are false.

Problem 6. Determine whether each of the following arguments is valid. If an argument is correct, what rule of inference (from Rosen, section 3.1) is being used? If it is not, what fallacy (from Rosen) occurs? If an argument is not correct, provide the correct argument.

(a) *Theorem:* if $n^2 > 1$, then $n > 1$. *Proof:* If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$, then $n > 1$.

(b) *Theorem:* The number $\log_2 3$ is irrational. *Proof:* A number is irrational if it is not the ratio of two integers. Therefore, since $\log_2 3$ cannot be written in the form a/b where a and b are integers, it is irrational.

(c) *Theorem:* If $n^2 \leq 4$ then, $n \leq 2$. *Proof:* Because if n is a real number with $n > 2$, then $n^2 > 4$.

(d) *Theorem* If $n \leq 3$, then $n^2 \leq 9$. *Proof:* Because if n is a real number with $n > 3$, then $n^2 > 9$.

Problem 7. Prove by contradiction: the square root of an irrational number is irrational.

Problem 8. Prove that $\sqrt[3]{5}$ is irrational.

Problem 9. Prove by cases that if x and y are real numbers, then

$$\max(x, y) = \frac{x + y}{2} + \frac{|x - y|}{2}$$

Problem 10. Using the axioms and inference rules for equality given in class, prove that

$$(\forall xyz)((x = y) \wedge (y = z)) \Rightarrow (x = z)$$