Corrections to Solution Set 5, Problem 6

As some of you might have noticed, there were some sign errors in the solution to problem 6 of problem set 5. Here is the corrected solution. I would like to thank Guy Fisher for pointing out the mistakes and supplying the corrections.

Problem 6

Given that $x < 1$ and $y < 1$, express the following in closed form

$$\prod_{p=1}^{\infty} \prod_{q=p}^{\infty} e^{-\frac{pq}{2} x^{p/2} y^q}$$

$$\prod_{p=1}^{\infty} \prod_{q=p}^{\infty} e^{-\frac{pq}{2} x^{p/2} y^q} = e^{-\sum_{p=1}^{\infty} \sum_{q=p}^{\infty} \frac{pq}{2} x^{p/2} y^q}$$

Now we compute $\sum_{p=1}^{\infty} \sum_{q=p}^{\infty} \frac{pq}{2} x^{p/2} y^q$.

$$\begin{aligned}
\sum_{q=p}^{\infty} \frac{pq}{2} x^{p/2} y^q &= \frac{p}{2} x^{p/2} \sum_{q=p}^{\infty} q y^q \\
&= \frac{p}{2} x^{p/2} \left(\sum_{q=1}^{\infty} q y^q - \sum_{q=1}^{p-1} q y^q \right) \\
&= \frac{p}{2} x^{p/2} y \left(\frac{p y^{p-1}}{1-y} + \frac{y^p}{(1-y)^2} \right)
\end{aligned}$$

We know from tutorial that

$$\sum_{k=1}^{\infty} k^2 a^k = \frac{2a^2}{(1-a)^3} + \frac{a}{(1-a)^2}$$

We will use this to compute the sum over p.

$$\begin{aligned}
\sum_{p=1}^{\infty}\sum_{q=p}^{\infty}\frac{pq}{2}x^{p/2}y^{q} &= \sum_{p=1}^{\infty}\frac{p}{2}x^{p/2}y\left(\frac{py^{p-1}}{1-y}+\frac{y^{p}}{(1-y)^{2}}\right)\\
&= \frac{1}{2(1-y)}\sum_{p=1}^{\infty}p^{2}(yx^{1/2})^{p}+\frac{y}{2(1-y)^{2}}\sum_{p=1}^{\infty}p(yx^{1/2})^{p}\\
&= \frac{1}{2(1-y)}\left(\frac{2xy^{2}}{(1-yx^{1/2})^{2}}+\frac{yx^{1/2}}{(1-yx^{1/2})^{2}}\right)+\frac{y}{2(1-y)^{2}}\frac{yx^{1/2}}{(1-yx^{1/2})^{2}}
\end{aligned}$$

Therefore the final result is

$$\prod_{p=1}^{\infty}\prod_{q=p}^{\infty}e^{-\frac{pq}{2}x^{p/2}y^{q}} = exp-\left(\frac{1}{2(1-y)}\left(\frac{2xy^{2}}{(1-yx^{1/2})^{2}}+\frac{yx^{1/2}}{(1-yx^{1/2})^{2}}\right)+\frac{y}{2(1-y)^{2}}\frac{yx^{1/2}}{(1-yx^{1/2})^{2}}\right)$$