Problem Set 11 Solutions

Problem 1. [10 points]

The couple who is having children until they get a boy and a girl knows that the expected number of children is 3. What is the variance?

After the first child, the waiting time until they get a success (a child of the opposite sex) is geometrically distributed. That is the probability that the first success occurs on the nth try is $(1-p)^{n-1}p$ ($n-1$ failures before 1 success).

The expectation of the geometric distribution is just the mean time to failure $\frac{1}{p}$.

The variance is given by

$$\sum_{n=1}^{\infty} n^2(1-p)^{n-1}p - \frac{1}{p^2}$$

From earlier in the course we know that

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3}$$

Setting $x = (1-p)$ we obtain

$\sum_{n=1}^{\infty} n^2(1-p)^{n-1}p = \frac{2}{p^2} - \frac{1}{p}$

Therefore the variance is $\frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{1}{p^2} - \frac{1}{p}$.

And putting $p = 1/2$ we get that the variance is 2.

Problem 2. [10 points]

Now suppose that the couple wants to get two girls in a row. Find the expected number of children.

This is the same problem as the one studied in lecture for building a system with n components. The components are the girls and boys are the failures in which case we have to start over (well, the couple has to start over).

We have $p = 1/2$, $n = 2$.

$\mathbf{E}[T] = \frac{1}{p(1-p)^n} - \frac{1}{p} = \frac{1}{1/8} - \frac{1}{1/2} = 6$

Problem 3. **[15 points]**

Let X and Y be random variables whose values are independent observations drawn uniformly at random from the set $\{1, \ldots, n\}$. Compute the following quantities:

(a) $\mathbf{Var}(aX + bY)$.

$$\mathbf{E}[X] = \frac{1}{n}\sum_{i=1}^{n} i = \frac{(n+1)}{2}$$

$$\mathbf{E}(X^2) = \frac{1}{n}\sum_{i=1}^{n} i^2 = \frac{1}{n}\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

so

$$\mathbf{Var}(X) = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}$$

$$\begin{aligned}
\mathbf{Var}(aX+bY) &= \mathbf{Var}(aX) + \mathbf{Var}(bY) \\
&= a^2\,\mathbf{Var}(X) + b^2\,\mathbf{Var}(Y) \\
&= (a^2+b^2)\,\mathbf{Var}(X) \\
&= \frac{(a^2+b^2)}{12}(n^2-1)
\end{aligned}$$

(b) $\mathrm{Ex}[\max(X, Y)]$.

Observe that $\mathbf{Pr}[\max(X,Y) = i] = \mathbf{Pr}[X = i \text{ and } Y < i] + \mathbf{Pr}[X < i \text{ and } Y = i] + \mathbf{Pr}[X = Y = i] = \frac{1}{n}\cdot\frac{i-1}{n} + \frac{i-1}{n}\cdot\frac{1}{n} + \frac{1}{n}\cdot\frac{1}{n} = \frac{2i-1}{n^2}$.
This means that $\mathbf{E}[\max(X,Y)] = \sum_{i=1}^{n}\frac{i(2i-1)}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n}(2i^2 - i) = \frac{1}{n^2}(\frac{2n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}) = \frac{(n+1)(4n-1)}{6n}$.

(c) $\mathbf{E}[\min(X,Y)]$. (*Hint:* There is a way to get the answer to this using the previous part.)

Observe that $\mathbf{E}[\max(X,Y)] + \mathbf{E}[\min(X,Y)] = \mathbf{E}[\max(X,Y) + \min(X,Y)] = \mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y]$. Now $\mathbf{E}[X] = \mathbf{E}[Y] = \frac{n+1}{2}$. so $\mathbf{E}[\min(X,Y)] = n + 1 - \frac{(n+1)(4n-1)}{6n} = \frac{(n+1)(2n+1)}{6n}$.

(d) $\mathbf{E}[|X - Y|]$

We have that $|X - Y| = \max(X,Y) - \min(X,Y)$, so by linearity of expectation $\mathbf{E}[|X-Y|] = \mathbf{E}[\max(X,Y)] - \mathbf{E}[\min(X,Y)] = \frac{(n+1)(4n-1)}{6n} - \frac{(n+1)(2n+1)}{6n} = \frac{(n^2-1)}{3n}$.

(e) $\mathbf{Var}(|X - Y|)$.

$\mathbf{Var}(|X-Y|) = \mathbf{E}[|X-Y|^2] - (\mathbf{E}[|X-Y|])^2$. But $\mathbf{E}[|X-Y|^2] = \mathbf{E}[(X-Y)^2] = \mathbf{Var}(X-Y) + (\mathbf{E}[X-Y])^2 = \mathbf{Var}(X-Y) = 2\,\mathbf{Var}(X) = \frac{n^2-1}{6}$ (since $\mathbf{E}[X-Y] = \mathbf{E}[X] - \mathbf{E}[Y] = 0$).

This means that $\mathbf{Var}[|X-Y|] = \frac{n^2-1}{6} - (\frac{n^2-1}{3n})^2 = \frac{(n^2-1)(n^2+2)}{18n^2}$.

Problem 4. **[10 points]**

(a) Let X be a random variable whose value is an observation drawn uniformly at random from the set $\{-n, -n+1, \ldots, -2, -1, 0, 1, 2, \ldots, n-1, n\}$. Let $Y = X^2$. Show that X and Y are *not* independent, but that $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$.

There are many examples to see that X and Y are not independent. An easy one is to note that $\mathbf{Pr}[X=2]\mathbf{Pr}[Y=1] \neq 0$ but $\mathbf{Pr}[XY=2] = \mathbf{Pr}[X^3=2] = 0$. To see that $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$, note that $\mathbf{E}[X] = 0$ and $\mathbf{E}[XY] = \mathbf{E}[X^3] = 0$ since X and X^3 are both symmetrical.

(b) Show in general for any two random variables X and Y that if $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$ then $\mathbf{Var}(X+Y) = \mathbf{Var}X + \mathbf{Var}Y$. In particular, this shows that it is possible for non independent variables to have variances that exhibit linearity.

We have that $\mathbf{Var}(X+Y) = \mathbf{E}[(X+Y)^2] - (\mathbf{E}[X+Y])^2 = \mathbf{E}[X^2 + 2XY + Y^2] - (\mathbf{E}[X] + \mathbf{E}[Y])^2 = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 + \mathbf{E}[Y^2] - (\mathbf{E}[Y])^2 + 2\mathbf{E}[XY] - 2\mathbf{E}[X]\mathbf{E}[Y] = \mathbf{Var}X + \mathbf{Var}Y$.

Problem 5. **[10 points]**

Consider the following 3-stage experiment:

In the first stage, I throw a 6-sided die. The number of spots that I get determines how many instances of the second stage to perform.

In the second stage, I toss a fair coin. This stage is repeated once for each spot I got in the first stage. The total number of heads that I get in this stage determines how many instances of the third stage to perform.

In the third stage, I throw a 6-sided die. This stage is repeated once for each head I got in all instances of the second stage.

Assume that all throws are mutually independent.

My score is the total number of spots on all the instances of the third stage.

What is my expected score?

Let X be the random variable whose value is the number of dots obtained in the first stage.

Let $Y_1, Y_2, \ldots, Y_X$ be the random variables that take the value of 1 if the coin comes up heads, and 0 otherwise for each coin toss in the second stage. Define $Y = Y_1 + \cdots + Y_X$.

Let $Z_1, Z_2, \ldots, Z_Y$ be the random variable whose values are the number of dots in the third stage. Define $Z = Z_1 + \cdots + Z_Y$. We want to find $\mathbf{E}[Z]$.

All the rolls and tosses are mutually independent, so we can use Wald's theorem to get $\mathbf{E}[Y] = \mathbf{E}[X]\mathbf{E}[Y_i] = (\frac{7}{2})(\frac{1}{2}) = \frac{7}{4}$.

Since Y is determined by tosses that are independent of the rolls in the third stage, Y and Z_i are independent. So we apply Wald's theorem again to get $\mathbf{E}[Z] = \mathbf{E}[Y]\mathbf{E}[Z_i] = (\frac{7}{2})(\frac{7}{4}) = \frac{49}{8}$.

The answer is $(\frac{49}{8})$.

Problem 6. [10 points]

Suppose that someone is infected with an unknown incurable contagious disease. Everyday he encounters n healthy people and infects each of them with probability p. Encounters are mutually independent. These people in turn meet n people and infect them with probability p and so on.

(a) Write a recurrence for the expected number of people on the $k+1$st day in terms of the expected number of infected people on the kth day. Assume that each day every sick person meets n *different* persons.

Let S_k be a random variable giving the number infected people after k days.
We can write a recurrence for S_{k+1} in terms of S_k.

Let the sick people after day k be numbered $1, \ldots, S_k$.
Let H_i be a random variable giving the number of healthy people infected by sick person i in one day.
Then H_i has a binomial distribution with parameters n and p.
Then:

$$S_{k+1} = S_k + \sum_{i=1}^{S_k} H_i$$

It follows from linearity of expectation that

$$\mathbf{E}[S_{k+1}] = \mathbf{E}[S_k] + \mathbf{E}[\sum_{i=1}^{S_k} H_i].$$

Then by Wald's theorem,

$$\mathbf{E}[S_{k+1}] = \mathbf{E}[S_k] + \mathbf{E}[S_k] \cdot \mathbf{E}[H]$$

where H is a variable with the binomial distribution with parameters n and p.
Thus, $\mathbf{E}[H] = np$.

So

$$\mathbf{E}[S_{k+1}] = \mathbf{E}[S_k] + \mathbf{E}[S_k] \cdot np = (1+np)\mathbf{E}[S_k]$$

Also, $\mathbf{E}[S_0] = 1$.
So we have a recurrence involving these expectations.
Writing e_i for $\mathbf{E}[S_i]$, we have:

$$e_{k+1} = (1+np)e_k$$

and

$$e_0 = 1$$

(b) Solve the recurrence of part (a).

$e_k = (1+np)^k$.

(c) Now suppose that any sick person recovers from the disease each day with probability r. Find the expected number of sick people on the kth day.

Let B_i be the indicator variable for whether the i^{th} sick person gets better. Then $\mathbf{E}[B_i] = r$. Then

$$S_{k+1} = S_k + \sum_{i=1}^{S_k} (H_i - B_i)$$

It follows that

$$\mathbf{E}[S_{k+1}] = \mathbf{E}[S_k] + \mathbf{E}[S_k] \cdot \mathbf{E}[H] - \mathbf{E}[S_k]\mathbf{E}[B_i]$$

and so

$$\begin{aligned} e_{k+1} &= e_k(1+np-r) \\ &= (1+np-r)^k \end{aligned}$$

Problem 7. [10 points]

A miner, lost in some abandoned mine, finds himself in a room containing three doors. Door 1 leads him to freedom after two hours travel; door 2 returns him to the same room after four hours travel, and door 3 returns him to the room after eight hours travel. Each time he finds himself in the room, the miner picks a door with equal probability. Let T denote the time until he finds the exit.

(a) Define a sequence of iid random variables X_i and a stopping rule N such that

$$T = \sum_{i=1}^{N} X_i$$

We view this as an experiment where at each step we make a choice between three equiprobable outcomes.

Let X_i = number of hours to return to room on ith step.

Let N = number of steps until good door is selected.

Then N is geometrically distributed with probability $1/3$.

(b) Use Wald's Identity to find $E[T]$

$\mathbf{Pr}(X_i = 2) = 1/3$
$\mathbf{Pr}(X_i = 4) = 1/3$
$\mathbf{Pr}(X_i = 8) = 1/3$
$\mathbf{E}[X_i] = 14/3$
$\mathbf{E}[N] = 3$
By Wald's Identity

$$\mathbf{E}[\sum_{i=1}^{N} X_i] = 14$$

Problem 8. [10 points]

Recall than in the Chinese Appetizer Problem R is number of people that get the same appetizer after a spin of the Lazy Susan.

(a) Compute the variance of R.

$\mathbf{Pr}(R = 0) = \frac{N-1}{N}$
$\mathbf{Pr}(R = N) = \frac{1}{N}$
$\mathbf{Var}(R) = \mathbf{E}[R^2] - \mathbf{E}[R]^2 = \frac{N^2}{N} - 1 = N - 1$

(b) Show that the upper bound that you get on $Pr(R \geq N)$ from the strong form of Chebyshev's inequality is tight.

The strong form of Chebyshev's inequality states that

$$\mathbf{Pr}(R - \mathbf{E}[R] \geq c \cdot \sigma_R) \leq \frac{1}{1 + c^2}$$

Set $c = \sqrt{N-1}$. Then $c \cdot \sigma_R = N - 1$ and

$$\mathbf{Pr}(R \geq N) = \mathbf{Pr}(R - 1 \geq N - 1) \leq \frac{1}{1 + (N-1)} = \frac{1}{N}$$

Since we know that $\mathbf{Pr}(R \geq N) = \frac{1}{N}$ for this problem, the bound is tight.

Problem 9. [10 points]

In this problem, we analyze probabilistically the 1st stage of bubble sort performed on a list of n integers $\{i_1, \ldots i_n\}$. We assume that all the integers are distinct.On the first step we compare integers i_1 and i_2. If $i_1 > i_2$ we swap them otherwise we leave them as they are. On step k $(2 \leq k \leq n-1)$, we compare the new i_k with i_{k+1}. If $i_k > i_{k+1}$ we swap else we leave them as they are. We assume that the order of the integers is chosen uniformly from all possible orderings.

(a) If a number x is randomly (uniformly) chosen from a set of n distinct numbers, what is the probability that x is the largest number in the set?

What is the probability that on the kth step $(2 \leq k \leq n-1)$, i_{k+1} is the biggest integer to appear in the list so far?

$\mathbf{Pr}(\text{choosing largest of } n \text{ numbers}) = 1/n$.

$\mathbf{Pr}(i_{k+1} \text{ in } k \text{ step is highest so far}) = 1/(k+1)$.

(b) Let C_k be a random variable that is equal to 1 if $i_{k+1} > i_k$. What is $\mathbf{E}[C_k]$?

$\mathbf{E}[C_k] = \mathbf{Pr}(C_k = 1) = \mathbf{Pr}(\ i_k \text{ is highest so far}) = \frac{1}{k+1}$.

(c) Let random variable $C = \sum_{k=1}^{n} C_k$. Give an asymptotic bound for $\mathbf{E}[C]$ as well as an interpretation of what $\mathbf{E}[C]$ is.

$\mathbf{E}[C] = \sum_k \mathbf{E}[C_k] = \sum_{k=2}^{n} \left(\frac{1}{k}\right) \sim \ln n$.

C is the total number of swaps, so $\mathbf{E}[C]$ is the expected number of swaps.