Problem Set 10 Solutions

Problem 1. [**10 points**]

In a physics experiment, every second a π-meson passes through a detector with probability p. π-meson arrivals are mutually independent from second to second.

(a) What is the probability of getting exactly k π-mesons passing through the detector in one minute?

At each second the appearance of a π-meson is the outcome of a Bernoulli trial with probability of success p. Therefore the number of π-mesons observed in 60 seconds is Binomially distributed.

$$\Pr(k) = \binom{60}{k} p^k (1-p)^{60-k}$$

(b) Suppose the detector is on for one minute. For what values of k is the probability of getting exactly k π-mesons greater than the probability of getting exactly $k-1$ π-mesons? For what values of k is it less? What value of k is the most likely number of π-mesons?

Consider the ratio of the probability of k π-mesons over the probability of $k-1$ π-mesons.

$$\begin{aligned}
\frac{\binom{60}{k} p^k (1-p)^{60-k}}{\binom{60}{k-1} p^{k-1} (1-p)^{60-k+1}} &= \frac{\frac{60!}{k!(60-k)!} p}{\frac{60!}{(k-1)!(60-k+1)!}(1-p)} \\
&= \frac{(60-k+1)p}{k(1-p)}
\end{aligned}$$

This quantity is greater or equal to 1 if and only if $(60-k+1)p \geq k(1-p) \iff 61p \geq k$. So, for $k \leq 61p$ the probability increases from $k-1$ to k, and for $k \geq 61p$, the probability decreases. The value of k that maximizes the probability is thus $\lfloor 61p \rfloor$. (Note: at the maximum value, the probability might change).

(c) Suppose that we keep observing until m π-mesons are observed. Find the probability distribution for the time T to see m π-mesons. What is the expected value of T?

Consider the event $\{T = l\}$. This means that there was one π-meson observed in the lth second, and there were $m-1$ π-meson arrivals on the previous $l-1$ seconds.

Since the arrivals at each second are independent, we have:

$$\begin{aligned}
\Pr(T = l) &= \Pr(m-1 \text{ arrivals in } l-1 \text{ seconds}) P(1 \text{ arrival on the } l\text{th second}) \\
&= \binom{l-1}{m-1} p^{m-1}(1-p)^{l-m} p \\
&= \binom{l-1}{m-1} p^{m}(1-p)^{l-m}
\end{aligned}$$

Let X be the time to observe one π-meson. Since future arrivals are independent of past arrivals, the time to see two π-mesons is $X+X = 2X$, and more generally the time to see m π-mesons is mX. Therefore $E[T] = mE[X] = \frac{m}{p}$.

Problem 2. [5 points]

A couple keeps having children until they have at least one boy and one girl. What is the expected number of children? Assume that boys and girls are equiprobable and that births are mutually independent.

The first birth is either a boy or a girl. Say it's a girl. Then they are having children until they get a boy. In class we showed that the expected number of children to get a boy is $1/\frac{1}{2} = 2$. So the total expected number of children until they get a boy and a girsl is $1+2 = 3$.

Problem 3. [10 points]

Remember the game demonstrated in lecture. Professor Leighton shows you two envelopes, each lying face down on a table. He tells you there's a number from the range $\{0, \ldots, n\}$ written in each. You pick an envelope, and Prof. Leighton shows you the number written in it. He now tells you that you can either choose to stay with that number, or switch to the number written in the other envelope. You *win* if the number you end up with is the biggest of the two (that is, if either you originally picked the biggest and decided to stay, or if you originally picked the smallest and decided to switch). A randomized strategy for this game is a function that, for each number i, assigns a probability p_i of switching *given* that the first card you picked was i. Prove that the strategy described in class for winning with probability better that $1/2$ is optimal. (i.e prove that if Prof. Leighton picks i and $i+1$ for the two numbers, where i is chosen uniformly in $[0, n-1]$, then your expected return is at most $\frac{1}{2} + \frac{1}{2n}$).

Let p_i denote the probability that you switch if you see the pair i. Then, the probability that you win if Prof. Leighton picks the pair $(i, i+1)$ is the probability that you will see the smaller number and switch or that you will see the bigger number and stay. Since the probability of picking either number is $1/2$, the probability that you win is

$\frac{1}{2}p_i + \frac{1}{2}(1 - p_{i+1})$.

Then, the total probability that you win is:

$$\sum_{i=0}^{n-1} \frac{1}{n}\left(\frac{1}{2}p_i + \frac{1}{2}(1-p_{i+1})\right) = \frac{1}{2} + \frac{1}{2n}\sum_{i=0}^{n-1}(p_i - p_{i+1}) = \frac{1}{2} + \frac{1}{2n}(p_0 - p_n) \leq \frac{1}{2} + \frac{1}{2n}$$

Problem 4. **[10 points]**

Now suppose the numbers from the problem above are drawn from the set of natural numbers, $\mathbb{N}$. Design a strategy for playing this game such that whatever numbers are picked by Prof. Leighton you will have more than an even chance of winning.

Let $p_i = 2^{-i-1}$ be the probability of guessing the number i. The strategy is to switch if the number you see is smaller than the one you guessed.

Suppose that the pair $(i, i+1)$ is chosen. Then the probability that you win is
$\frac{1}{2}p_i + \frac{1}{2}(1-p_{i+1}) = \frac{1}{2} + \frac{1}{2}(2^{-i-1} - 2^{-i-2}) = \frac{1}{2} + \frac{1}{4}2^{-i-1} > \frac{1}{2}$

Problem 5. **[7 points]**

Show that if two positive random variables R and T are independent, then $\sqrt{R}$ and $\log T$ are also independent.

$$\begin{aligned} Pr((\sqrt{R} = r) \cap (\log T = t)) &= Pr[(R = r^2) \cap (T = e^t)] \\ &= Pr(R = r^2)Pr(T = e^t) \\ &= Pr(\sqrt{R} = r)Pr(\log T = t) \end{aligned}$$

Problem 6. **[10 points]**

(a) [5 points]

Let R_1 and R_2 be two independent random variables. R_1 takes the values $0, 1$ with equal probability. R_2 has the following distribution.
$P(R_2 = 1) = 1/2$
$P(R_2 = 3) = 1/4$
$P(R_2 = 4) = 1/4$
Compute the distribution of $R_1 2^{R_2}$.

Let $X = R_1 2^{R_2}$.
$\Pr(X = 0) = \Pr(R_1 = 0) = 1/2$.
$\Pr(X = 2) = \Pr(R_1 = 1 \wedge R_2 = 1) = 1/4$.
$\Pr(X = 8) = 1/8$.
$\Pr(X = 16) = 1/8$.

(b) [5 points]

Now suppose that R_1 and R_2 are no longer independent. In particular $R_1 = 0$ if R_2 is even, and $R_1 = 1$ when R_2 is odd. R_2 has the same distribution as in part (a).

Compute the distribution of $R_1 2^{R_2}$

$\Pr(X = 0) = \Pr(R_2 \text{ is even}) = 1/4$.
$\Pr(X = 2) = \Pr(R_2 = 1) = 1/2$.
$\Pr(X = 8) = \Pr(R_2 = 3) = 1/4$.

Problem 7. [10 points]

Three gamblers meet at a bar. Each wagers 1 dollar and the bartender flips a fair coin. The money is shared among the players who guessed correctly the outcome of the coin. If no one guesses correctly, each player gets his money back.

(a) The coin is fair and no player sees the outcome before guessing. What is the expected return for each player if each player assumes that the other players guess heads independently with probability 1/2?

Let X be the expected return of a player. It can take the values $-1, 0, 0.5, 2$ with the following probabilities:

$P(X = -1) = (1/2)(3/4)$ (half a chance that he will guess wrong and (3/4) the others will guess right)
$P(X = 0) = (1/8) + (1/8)$ (everybody will guess correctly or everybody will guess wrong)
$P(X = 0.5) = (1/2)(1/2)$ (one other player will guess correctly)
$P(X = 2) = (1/2)(1/4)$ (only he will guess correctly)

Therefore $E[X] = (3/8)(-1) + (1/4)0 + (1/4)(0.5) + (1/8)2 = 0$.

(b) Show that the answer you got in part (a) is not always correct if we don't assume that the player's choices are mutually independent. (i.e show how two players can cooperate to improve their expected return).

Suppose players B and C decide to bet always the opposite (one bets head the other bets tail). Then X_A, the expected return of player A, can take only the values -1 and 0.5 with probability 1/2 each, so the expected return is -0.25.

On the other hand the expected return for either of the other players, say X_B has the following distribution

$P(X_B = -1) = (1/2)$ (probability that he will guess wrong. In this case C guesses right).
$P(X_B = 0) = 0$ (It's never the case that they all guess the same outcome).
$P(X_B = 0.5) = (1/2)(1/2)$ (He and A will guess correctly)

$P(X_B = 2) = (1/2)(1/2)$ (He will guess correctly and A wrong).

$E[X_B] = (1/2)(-1) + (1/4)0.5 + (1/4)2 = 0.125$

Problem 8. **[10 points]**

In a recent column Marilyn analyzed the following game:

A fair coin is flipped until you get heads.

You pay 2^{i-1} dollars if the first head appears in the ith toss. What is the expected amount paid?

Marilyn argued that the expected amount is infinite because you might get very unlucky and pay a lot.

In this problem you will demonstrate that Marilyn's claim is right but her "proof" is wrong.

(a) Show that the expected payment for the game described is indeed infinite.

The probability that the 1st head appears in the ith toss is $(1/2)^i$,therefore the expected payment is

$$\sum_{i=1}^{\infty}(1/2)^i 2^{i-1} = \sum_{i=1}^{\infty}(1/2) = \infty$$

(b) Design a variant of the game (by modifying the amount paid while keeping it exponential) in which Marylin's claim is not true.
Let the amount be $(3/2)^{i-1}$.
The expected payment is

$$\sum_{i=1}^{\infty}(1/2)^i(3/2)^{i-1} = (2/3)\sum_{i=1}^{\infty}(3/4)^i = 2$$

Problem 9. **[5 points]**

Flip a fair coin until you get heads. What's the probability that you will see an even number of tails? Assume that the coin tosses are mutually independent.

The probability that the 1st head appears on the ith toss is $\left(\frac{1}{2}\right)^i$.

$$\begin{aligned} \Pr(\text{even tails}) &= \sum_{i \text{ is odd}} (1/2)^i \\ &= \sum_{i=0}^{\infty}(1/2)^{2i+1} \end{aligned}$$

$$
\begin{aligned}
&= (1/2)\sum_{i=0}^{\infty}(1/4)^i \\
&= 2/3
\end{aligned}
$$

Problem 10. **[5 points]**

We want to determine the percentage r of people that prefer Ben and Jerry's Chocolate Fudge Brownie over Toscannini's Cherry Garcia. How many people should be polled so that the result is within 1 point of the true percentage with probability at least .95?

$\epsilon = .01, \delta = .05$

$$
\begin{aligned}
\delta \geq 2F_{1/2}((1/2-\epsilon)n) \quad &\Rightarrow \quad .05 \geq 2F_{1/2}(0.49n) \\
&\Rightarrow \quad 0.025 \geq \frac{0.51}{0.02} f_{1/2}(0.49n) \\
&\Rightarrow \quad 0.025 \geq \frac{25.5}{\sqrt{2n\pi(.49)(.51)}} 2^{(H(.49)-1)n} \\
&\Rightarrow \quad 0.0012285 \geq \frac{2^{(H(.49)-1)n}}{\sqrt{n}} \\
&\Rightarrow \quad 0.0012285 \geq \frac{2^{-0.000288574n}}{\sqrt{n}} \\
&\Rightarrow \quad 0.0012285 \geq \frac{(.9998)^n}{\sqrt{n}}
\end{aligned}
$$

Choosing $n = 10388$ will do it.

Problem 11. **[10 points]**

Let D_1 and D_2 denote independent random variables uniformly distributed in $\{1, \ldots, n\}$. Let $M_1 = \max(D_1, D_2)$, and $M_2 = \min(D_1, D_2)$.

(a) Compute the distribution function for M_1.

$$
\begin{aligned}
\Pr(M_1 = x) &= \Pr(\max(D_1, D_2) = x) \\
&= \Pr(D_1 = x \text{ and } D_2 < x) + \Pr(D_1 < x \text{ and } D_2 = x) + \Pr(D_1 = D_2 = x) \\
&= \frac{1}{n} \cdot \frac{x-1}{n} + \frac{x-1}{n} \cdot \frac{1}{n} + \frac{1}{n} \cdot \frac{1}{n} \\
&= \frac{2x-1}{n^2}
\end{aligned}
$$

(b) Are M_1 and M_2 independent random variables? Justify your answer.

M_1 and M_2 are not independent since we have:

$$\begin{aligned} \Pr(M_1 = x | M_2 > x) &= 0 \qquad \text{for } x \in \{1, 2, \ldots n\} \\ &\neq \Pr(M_1 = x) \end{aligned}$$

Problem 12. **[8 points]**

Consider a system consisting of 2 components, each of which can fail at each step with probability p. Failures at different steps and/or components are mutually independent (in particular, this means that failure need not be permanent). The steps are numbered $1, 2, \ldots$.

(a) **[5 points]**
Let X be the random variable whose value is the step number of the first step at which there is a failure (in either component). Find $E[X]$.

The probability that some of the components fail is $1 - (1-p)^2 = 2p - p^2$.
The distribution of X is
$\Pr(X = i) = (1-p)^{2(i-1)}(2p - p^2)$.
From class, the mean time to failure is $\frac{1}{2p-p^2}$.

(b) **[5 points]**
Let Y be the random variable whose value is the step number of the first step at which both components fail simultaneously. Find $E[Y]$.

The probability that both components will fail simultaneously is p^2.
The distribution of Y is
$\Pr(Y = i) = (1-p^2)^{i-1}p^2$.
so the mean time to failure is $\frac{1}{p^2}$.