Problem Set 11

Due: December 5 , 1997 in tutorial.

Reading: Lecture notes

Problem 1. [**10 points**]

The couple who is having children until they get a boy and a girl knows that the expected number of children is 3. What is the variance?

Problem 2. [**10 points**]

Now suppose that the couple wants to get two girls in a row. Find the expected number of children.

Problem 3. [**15 points**]

Let X and Y be random variables whose values are independent observations drawn uniformly at random from the set $\{1, \ldots, n\}$. Compute the following quantities:

(a) $\mathbf{Var}(aX + bY)$.

(b) $\mathbf{E}[\max(X, Y)]$.

(c) $\mathbf{E}[\min(X, Y)]$. (*Hint:* There is a way to get the answer to this using the previous part.)

(d) $\mathbf{E}[|X - Y|]$.

(e) $\mathbf{Var}(|X - Y|)$.

Problem 4. [**10 points**]

(a) Let X be a random variable whose value is an observation drawn uniformly at random from the set $\{-n, -n+1, \ldots, -2, -1, 0, 1, 2, \ldots, n-1, n\}$. Let $Y = X^2$. Show that X and Y are *not* independent, but that $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$.

(b) Show in general for any two random variables X and Y that if $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$ then $\mathbf{Var}(X + Y) = \mathbf{Var}X + \mathbf{Var}Y$. In particular, this shows that it is possible for non independent variables to have variances that exhibit linearity.

Problem 5. **[10 points]**

Consider the following 3-stage experiment:

In the first stage, I throw a 6-sided die. The number of spots that I get determines how many instances of the second stage to perform.

In the second stage, I toss a fair coin. This stage is repeated once for each spot I got in the first stage. The total number of heads that I get in this stage determines how many instances of the third stage to perform.

In the third stage, I throw a 6-sided die. This stage is repeated once for each head I got in all instances of the second stage.

Assume that all throws are mutually independent.

My score is the total number of spots on all the instances of the third stage.

What is my expected score?

Problem 6. **[15 points]**

Suppose that someone is infected with an unknown incurable contagious disease. Everyday he encounters n healthy people and infects each of them with probability p. Encounters are mutually independent. These people in turn meet n people and infect them with probability p and so on.

(a) Write a recurrence for the expected number of people on the $k+1$st day in terms of the expected number of people on the kth day. Assume that each day every sick person meets n *different* persons.

(b) Solve the recurrence of part (a).

(c) Now suppose that any sick person recovers from the disease each day with probability r. Find the expected number of sick people on the kth day.

Problem 7. **[10 points]**

A miner, lost in some abandoned mine, finds himself in a room containing three doors. Door 1 leads him to freedom after two hours travel; door 2 returns him to the same room after four hours travel, and door 3 returns him to the room after eight hours travel. Each time he finds himself in the room, the miner picks a door with equal probability. Let T denote the time until he finds the exit.

(a) Define a sequence of iid random variables X_i and a stopping rule N such that

$$T = \sum_{i=1}^{N} X_i$$

(b) Use Wald's Identity to find $\mathbf{E}[T]$

Problem 8. **[10 points]**

Recall than in the Chinese Appetizer Problem R is number of people that get the same appetizer after a spin of the Lazy Susan.

(a) Compute the variance of R.

(b) Show that the upper bound that you get on $\mathbf{Pr}(R \geq N)$ from the strong form of Chebyshev's inequality is tight.

Problem 9. **[10 points]**

In this problem, we analyze probabilistically the 1st stage of bubble sort performed on a list of n integers $\{i_1, \ldots i_n\}$. We assume that all the integers are distinct. On the first step we compare integers i_1 and i_2. If $i_1 > i_2$ we swap them otherwise we leave them as they are. On step k $(2 \leq k \leq n-1)$, we compare the new i_k with i_{k+1}. If $i_k > i_{k+1}$ we swap else we leave them as they are. We assume that the order of the integers is chosen uniformly from all possible orderings.

(a) If a number x is randomly (uniformly) chosen from a set of n distinct numbers, what is the probability that x is the largest number in the set?

What is the probability that on the kth step $(2 \leq k \leq n-1)$, i_{k+1} is the biggest integer to appear in the list so far?

(b) Let C_k be a random variable that is equal to 1 if $i_{k+1} > i_k$. What is $\mathbf{E}[C_k]$?

(c) Let random variable $C = \sum_{k=1}^{n} C_k$. Give an asymptotic bound for $\mathbf{E}[C]$ as well as an interpretation of what $\mathbf{E}[C]$ is.