Problem Set 9 Solutions

Problem 1. [**5 points**]

Consider playing the *Lets Make a Deal*, where there are three prizes: a "Gift Certificate" from Toscanini, a "Nerd Pocket Protector", and a "Nerd Pride Button". Your goal is to win the gift certificate.

(a) Suppose you (the contestant) happen to know that Carol always reveals the "Nerd Pocket Protector", if you did not choose the box covering it.

Does this additional information change the odds of winning for you? Is there a better strategy? By how much (if any) does this information increase your chances of winning?

The probability of winning the game remains the same but now there is more than one optimal strategy. If Carol reveals the "Nerd Pride Button" it means that you must have chosen the box covering the "Nerd Pocket Protector". In this case, you win with probability 1 if you switch. If Carole shows you the "Nerd Pocket Protector" then there is 1/2 probability that you have picked the prize box, so you win with probability 1/2 whether you switch or not. Let NPB be the event that Carole shows the "Nerd Pride Button", and NPP be the event that Carole shows the "Nerd Pocket Protector". Then the total probability of winning is:

$$\begin{aligned} Pr(win) &= Pr(win \mid \mathrm{NPB})Pr(\mathrm{NPB}) + Pr(win \mid \mathrm{NPP})Pr(\mathrm{NPP}) \\ &= 1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} \\ &= \frac{2}{3} \end{aligned}$$

(b) You are playing according to the original assumptions, but before you make your initial choice, you happen to see Carol placing a "Nerd Pride Button" underneath Box 1. What should you do to try to win the prize (the ice cream)? What is the probability that you win?

In this case, you should always pick Box 1. Monty will be forced to reveal the location of the other "Nerd Pride" pin. Upon switching, you will be guaranteed to win the ice cream (i.e. the probability of winning is 1). If you pick a box besides 1, Monty can always reveal Box 1 (giving you no additional information), and you will have a 1/2 chance of winning.

Problem 2. **[10 points]**

Consider a three by three array of squares as in the game of tic-tac-to.

(a) If three squares are picked at random (independently and with the same probability), what is the probability that these squares will form a line? (horizontal, vertical, or diagonal)

There are $\binom{9}{3}$ ways to choose three squares in the grid. There are three ways three squares can form a horizontal line, three ways they can form a vertical line, and two ways they can form a diagonal line. So the probability that three randomly selected squares form a line is

$\frac{8}{\binom{9}{3}} = \frac{2}{21}$

(b) If you pick four squares at random, what is the probability that three of them will be on a line?

There are $\binom{9}{4}$ ways to pick the squares. For each line formed there are 6 ways to choose the position of the fourth square, so there are 48 that the squares will form a line. The probability is

$\frac{48}{\binom{9}{4}} = \frac{8}{21}$.

Problem 3. **[10 points]**

(a) You roll two dice. What is the probability that their sum is not 6?

There are five ways the sum can be 6: $(1,5),(5,1),(2,4),(4,2),(3,3)$

There are $6 \cdot 6 = 36$ ways the dice can land, so the probability that their sum is six is $\frac{5}{36}$. Therefore the probability that the sum is not six is $\frac{31}{36}$.

(b) You roll three dice. What is the probability that that any of the *pairwise* sums is not 6?

There are $6^3 = 216$ possible dice rolls. Define the events:

$A_1 = \{d_1 + d_2\} = 6$

$A_2 = \{d_2 + d_3\} = 6$

$A_3 = \{d_3 + d_1\} = 6$

For each A_i, $P(A_i) = 5/36$.

We also have $P(A_i \cap A_j) = 5/216$, while $P(A_1 \cap A_2 \cap A_3) = 1/216$.

So

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= 3P(A_i) - 3P(A_i \cap A_j) + P(A_1 \cap A_2 \cap A_3) \\ &= \frac{90 - 15 + 1}{216} = \frac{76}{216} \end{aligned}$$

so the probability of not getting a six is 140/216.

Problem 4. **[10 points]**

(a) Prove that $Pr(\overline{A}|C) = 1 - Pr(A|C)$ for any events A and C.

$$\begin{aligned} 1 - Pr(A|C) &= 1 - \frac{Pr(A \cap C)}{Pr(C)} \\ &= \frac{Pr(C) - Pr(A \cap C)}{Pr(C)} \\ &= \frac{Pr(\overline{A} \cap C)}{Pr(C)} \\ &= Pr(\overline{A}|C) \end{aligned}$$

(b) Is it possible for $Pr(A|B)$ and $Pr(A|C)$ to be very small, if $Pr(A|B \cap C)$ is large? Give a counterexample or a proof to justify your answer.

Yes, it is. Consider the uniform sample space $S = \{1, 2, \ldots, 2 \cdot 10^6\}$.
Let $A = \{1\}, B = \{1, 2, \ldots, 10^6\}, C = \{1, 10^6 + 1, \ldots, 2 \cdot 10^6 - 1\}$.
Then $Pr(A \mid B) = Pr(A \mid C) = 10^{-6}$, while $Pr(A \mid B \cap C) = 1$.

Problem 5. **[5 points]**

Three Prisoners A,B, and C have applied for parole. The parole board has decided to randomly choose two of the three to be released, and the prisoners know this but not which two. A warder friend of prisoner A knows who are to be released. A realizes that it would be unethical to ask if he is going to be released, but thinks of asking for the name of *one prisoner other than himself* who is to be released. Before he asks his chances are 2/3, but he thinks that if he asks and the warder says "B will be released", his own chances will become 1/2 since either A or C will be the only remaining candidates. So A decides not to reduce his chances by asking. Is A's argument correct? Show what his chances are after asking. You may assume that if the warder has a choice of which name to reveal, then he picks one at random.

Let W_B be the event that the Warder says B. From picture 1 we see that $P(W_B) = (1/3)(1/2) + (1/3)1 = 1/2$.

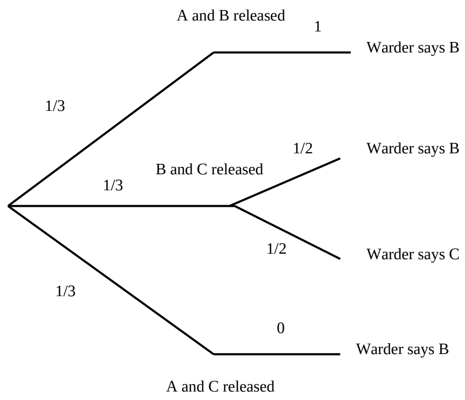

Figure 1: Probability that Warder says B

$A \cap W_B$ is the event that A is released and the warder says B, which is the same as the event that A and B are released, so $P(A \cap W_B) = 1/3$.

$P(A|W_B) = \frac{P(A \cap W_B)}{P(W_B)} = 2/3$ so the chances of A are not reduced by asking after all.

Problem 6. **[10 points]**

We know from statistics that when a policeman gets shot he survives with probability 0.8 if he/she wears a bullet-proof vest and survives with probability 0.12 if the policeman doesn't wear a vest. Given that 15% of the policemen wear bullet-proof vests, what is the probability that a random policeman was wearing a bullet-proof vest if he/she survived a firearm attack?

Let B be the event the policeman wears a vest. Let W be the event the policeman survives an attack. We want to compute $Pr(B|W)$. By Bayes' theorem,

$$Pr(B|W) = \frac{Pr(W|B)Pr(B)}{Pr(W|B)Pr(B) + Pr(W|\overline{B})Pr(\overline{B})} = \frac{(0.8)(.15)}{(0.8)(0.15) + (0.12)(0.85)} = 0.54$$

Problem 7. **[10 poiints]**

A, B, and C are to fight a three-cornered pistol duel. They all know that A's chance of hitting the target is 0.3, C's is 0.5 and B never misses. They are to shoot in the sequence A, B, C and fire at their choice of target until one person is left standing. Show that A is better off missing deliberately the first shot than attempting to hit either B or C.

First consider a scenario where only A and C are left. Let S be the event that A survives if C shoots first.

$$P(S) = 0.5(0.3 + 0.7 \cdot P(S)) \Rightarrow P(S) = 0.23$$

Let's calculate A's chances of survival under the three possible strategies.

1. If A shoots B first, then with probability 0.3 he kills him and then we are in the above scenario. If A misses B, then B kills C and than A has 0.3 chance to hit B and survive. So A's probability of survival under this strategy is $.3 * P(S) + .7 * .3 = .28$.
2. If A shoots C and kills him then he will be killed by B, while if he misses, B will kill C and then A has 0.3 probability to get B So A's probability of survival is $0.7 * .3 = 0.21$.
3. If A misses deliberately then B will kill C and A has probability 0.3 to survive.

So A best strategy is to shoot in the air. Under either strategy, A's chances are slim. I wonder how he got himself into that.

Problem 8. **[10 points]**

(a) Show that if two events A and B are independent, then the events A and $\overline{B}$ are also independent.

A and B independent means $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$. We have to show $\Pr(A \cap \overline{B}) = \Pr(A) \cdot \Pr(\overline{B})$. Since A is the disjoint union of $(A \cap B)$ and $(A \cap \overline{B})$ we have

$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap \overline{B})$$

and therefore

$$\begin{aligned}
\Pr(A \cap \overline{B}) &= \Pr(A) - \Pr(A \cap B) \\
&= \Pr(A) - \Pr(A)\Pr(B) \\
&= \Pr(A)(1 - \Pr(B)) \\
&= \Pr(A)\Pr(\overline{B})
\end{aligned}$$

proving that A and $\overline{B}$ are independent.

(b) Two numbers x and y are selected at random from the set $\{1, \ldots, 100\}$. Let the events A, B and C be defined as follows: $A = \{x > 50\}$, $B = \{y > 50\}$ and $C = \{x > y\}$. Are the events A and B independent? Are A and C independent?

First of all we compute $\Pr(A) = \Pr(B) = 50/100 = 0.5$ and $\Pr(C) = \binom{100}{2} \cdot 100^{-2} = 0.495$.
Since $\Pr(A \cap B) = (50 \cdot 50)/100^2 = 0.25 = \Pr(A) \cdot \Pr(B)$, A and B are independent.

$$\begin{aligned}
\Pr(A \cap C) &= \Pr(x > y \cap x > 50) \\
&= \sum_{i=51}^{100} \Pr(x > y \cap x = i)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=51}^{100} \Pr(y < i) \Pr(x = i) \\
&= (1/100) \sum_{i=51}^{100} \Pr(y < i) \\
&= (1/100) \sum_{i=51}^{100} \frac{i-1}{100} \\
&= (1/100)((\frac{100 \cdot 101}{2} - \frac{50 \cdot 51}{2})(1/100) - 49/100) \\
&= 0.3726
\end{aligned}$$

while $\Pr(A)\Pr(C) = .247$. So A and C are not independent.

Problem 9. **[10 points]**

You are given a coin of bias p (prob of heads = p; tails = 1-p, $0 < p < 1$). The value of p is not known and you are not allowed to guess by sampling

Using this, how can you get a coin of bias:

1. p^2

   Flip twice. Let B be the event that head appears both times.

2. $\frac{1}{2}p^2 + \frac{1}{4}p$

   Here's how to make $\frac{1}{2}$. Flip twice. If the result is HT you win. If it's TH you loose (note that each of these events occurs with probability $p(1-p)$. If it's something else you repeat the experiment.
   Let P_W denote the probability that you win. Then P_W obtains the recursion
   $P_W = p(1-p) + (1 - 2p(1-p))P_W$
   Solving for P_W we get $P_W = 1/2$.
   Let B be the event that you win. We will refer to the experiment described as the "fair experiment"
   To form $\frac{1}{2}p^2 + \frac{1}{4}p$ conduct the fair experiment. If you win, flip twice. If both flips are heads you win. If you lose in the fair experiment, repeat the fair experiment again and if you succeed flip the coin. If you get heads you win. If the fair experiment fails again or if you get tails you loose.
   Let B be the event that you win.
   Now we show that $P(B) = \frac{1}{2}p^2 + \frac{1}{4}p$. Let F be the event that the fair experiment succeeds, and H the event that the coin toss is heads.

$$\begin{aligned}
P(B) &= P(B|F)P(F) + P(B|\bar{F})P(\bar{F}) \\
&= \frac{1}{2}p^2 + \frac{1}{2}P(B|\bar{F})
\end{aligned}$$

Now $P(B|\bar{F}) = P(\text{both fair experiment and coin toss succed}) = P(F \cap H) = \frac{1}{2}p$. So,

$$P(B) = \frac{1}{2}p^2 + \frac{1}{2}\frac{1}{2}p = \frac{1}{2}p^2 + \frac{1}{4}p$$

3. $\frac{1}{2} - \frac{1}{4}p$

   Note that $\frac{1}{2} - \frac{1}{4}p = 1 - \left(\frac{1}{2} + \frac{1}{4}p\right)$.

   Perform the fair experiment. If it succeeds you win. If it fails, perform the fair experiment again. If it fails you lose. If it succeeds you flip the coin and win if you get heads.

   Let B be the event that you lose.

   To prove that this works, let's calculate $P(\bar{B})$.

$$\begin{aligned} P(\bar{B}) &= P(\bar{B}|F)P(F) + P(\bar{B}|\bar{F})P(\bar{F}) \\ &= \frac{1}{2} + \frac{1}{2}P(H) \\ &= \frac{1}{2} + \frac{1}{2}p \end{aligned}$$

Problem 10. **[10 points]**

Consider three loaded dice such that

Die A always comes up 3
Die B is 2 with prob p and 5 otherwise
Die C is 4 with prob p and 1 otherwise.

Find a value for p such that A beats B, B beats C, and C beats A all with probability p.

It's clear that A beats B and C beats A with probability p. In order for B to beat C with probability C we must have:

$$(1-p) + p(1-p) = p \Rightarrow p^2 + p - 1 = 0$$

Solving this quadratic (and observing the restriction $0 \leq p \leq 1$), we obtain $p = \frac{\sqrt{5}-1}{2}$.

Problem 11. **[10 points]**

In a complex system with 100,000 different components. The system contains 10 critical components, each failing with probability 1/1000 and 100 secondary components each failing with probability 1/100. It is known that secondary failures are pairwise independent. The system crashes if one critical component or two secondary components fail.

(a) Determine an upper bound on the probability that the system will crash.

Let C_i be the event that critical component i fails. Let S be the event that two secondary components fail. We know that $P(C_i) = 1/1000$. There are $100^2 = 10000$ ways that to distribute failures among the secondary components, and $\binom{100}{2}$ ways to choose the two components that fail, so the probability that two components will fail is $\frac{\binom{100}{2}}{10000} = 99/200$.

$$P(\text{system crashes}) = P(\cup_{i=1}^{10} C_i \cup S) \le \sum_{i=1}^{10} P(C_i) + P(S) = .01 + .495 = .505$$

(b) Determine a lower bound on the probability that the system will crash.

In the best possible case all the critical components will fail if and only if a critical component fails. In this case the probability that the system crashes is at least 1/1000. We will show that we can make it exactly 1/1000 as follows.

Suppose that 1/10 of the time that a critical component fails, all the secondary component's will fail. (Thus all the secondary components fail with probability 1/10000).

Now let the probability of a a secondary component failing *conditioned* on the event that no critical component fails be 99/9990 and make these events disjoint (so that no two secondary components can fail conditioned on the event that no critical component fails).

We have

$P(\text{A critical component fails}) = 1/1000$

$P(\text{A pair of secondary component fails}) = P(\text{All secondary component fail}) = 1/10000$

Let S_i be the event that the ith secondary component fails. Let C be the event that some critical component (and therefore all of them) fails.

$$\begin{aligned}
P(S_i) &= P(S_i \cap C) + P(S_i \cap \bar{C}) \\
&= 1/1000 + P(S_i|\bar{C})P(\bar{C}) \\
&= 1/10000 + (99/9990) \cdot (999/1000) \\
&= 1/10000 + (99/10000) \\
&= 1/100
\end{aligned}$$

So the system specifications are met and the probability that the system will crash is 1/1000.

(c) Where is the biggest weakness? In the critical, or in the secondary components?

In the secondary components. The probability that a primary component fails is bounden above by .01 while the probability that some two secondary components will fail is almost 1/2.