Problem Set 10

Due: November 25, 1997.

Reading: Lecture notes

Problem 1. **[10 points]**

In a physics experiment, every second a π-meson passes through a detector with probability p. π-meson arrivals are mutually independent from second to second.

(a) What is the probability of getting exactly k π-mesons passing through the detector in one minute?

(b) Suppose the detector is on for one minute. For what values of k is the probability of getting exactly k π-mesons greater than the probability of getting exactly $k-1$ π-mesons? For what values of k is it less? What value of k is the most likely number of π-mesons?

(c) Suppose that we keep observing until m π-mesons are observed. Find the probability distribution for the time T to see m π-mesons. What is the expected value of T?

Problem 2. **[5 points]**

A couple keeps having children until they have at least one boy and one girl. What is the expected number of children? Assume that boys and girls are equiprobable and that births are mutually independent.

Problem 3. **[10 points]**

Remember the game demonstrated in lecture. Professor Leighton shows you two envelopes, each lying face down on a table. He tells you there's a number from the range $\{0, \ldots, n\}$ written in each. You pick an envelope, and Prof. Leighton shows you the number written in it. He now tells you that you can either choose to stay with that number, or switch to the number written in the other envelope. You *win* if the number you end up with is the biggest of the two (that is, if either you originally picked the biggest and decided to stay, or if you originally picked the smallest and decided to switch). A randomized strategy for this game is a function that, for each number i, assigns a probability p_i of switching *given* that the first card you picked was i. Prove that the strategy described in class for winning with probability better that 1/2 is optimal. (i.e prove that if Prof. Leighton picks i and $i+1$ for the two numbers, where i is chosen uniformly in $[0, n-1]$, then your expected return is at most $\frac{1}{2} + \frac{1}{2n}$).

Problem 4. **[10 points]**

Now suppose the numbers from the problem above are drawn from the set of natural numbers, $\mathbb{N}$. Design a strategy for playing this game such that whatever numbers are picked by Prof. Leighton you will have more than an even chance of winning.

Problem 5. **[7 points]**

Show that if two positive random variables R and T are independent, then $\sqrt{R}$ and $\log T$ are also independent.

Problem 6. **[10 points]**

(a) **[5 points]**

Let R_1 and R_2 be two independent random variables. R_1 takes the values $0, 1$ with equal probability. R_2 has the following distribution.

$P(R_2 = 1) = 1/2$
$P(R_2 = 3) = 1/4$
$P(R_2 = 4) = 1/4$

Compute the distribution of $R_1 2^{R_2}$.

(b) **[5 points]**

Now suppose that R_1 and R_2 are no longer independent. In particular $R_1 = 0$ if R_2 is even, and $R_1 = 1$ when R_2 is odd. R_2 has the same distribution as in part (a).

Compute the distribution of $R_1 2^{R_2}$

Problem 7. **[10 points]**

Three gamblers meet at a bar. Each wagers 1 dollar and the bartender flips a fair coin. The money is shared among the players who guessed correctly the outcome of the coin. If no one guesses correctly, each player gets his money back.

(a) The coin is fair and no player sees the outcome before guessing. What is the expected return for each player if each player assumes that the other players guess heads independently with probability 1/2?

(b) Show that the answer you got in part (a) is not always correct if we don't assume that the player's choices are mutually independent. (i.e show how two players can cooperate to improve their expected return).

Problem 8. **[10 points]**

In a recent column Marilyn analyzed the following game:

A fair coin is flipped until you get heads.

You pay 2^{i-1} dollars if the first head appears in the ith toss. What is the expected amount paid?

Marilyn argued that the expected amount is infinite because you might get very unlucky and pay a lot.

In this problem you will demonstrate that Marilyn's claim is right but her "proof" is wrong.

(a) Show that the expected payment for the game described is indeed infinite.

(b) Design a variant of the game (by modifying the amount paid while keeping it exponential) in which Marylin's claim is not true.

Problem 9. **[5 points]**

Flip a fair coin until you get heads. What's the probability that you will see an even number of tails? Assume that the coin tosses are mutually independent.

Problem 10. **[5 points]**

We want to determine the percentage r of people that prefer Ben and Jerry's Chocolate Fudge Brownie over Toscannini's Cherry Garcia. How many people should be polled so that the result is within 1 point of the true percentage with probability at least .95?

Problem 11. **[10 points]**

Let D_1 and D_2 denote independent random variables uniformly distributed in $\{1, \ldots, n\}$. Let $M_1 = max(D_1, D_2)$, and $M_2 = min(D_1, D_2)$.

(a) Compute the distribution function for M_1.

(b) Are M_1 and M_2 independent random variables? Justify your answer.

Problem 12. **[8 points]**

Consider a system consisting of 2 components, each of which can fail at each step with probability p. Failures at different steps and/or components are mutually independent (in particular, this means that failure need not be permanent). The steps are numbered $1, 2, \ldots$.

(a) **[5 points]**
Let X be the random variable whose value is the step number of the first step at which there is a failure (in either component). Find $\text{Ex}[X]$.

(b) **[5 points]**
Let Y be the random variable whose value is the step number of the first step at which both components fail simultaneously. Find $\text{Ex}[Y]$.