Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professors Tom Leighton and Santosh Vempala

Handout 27
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Problem Set 8 Solutions

Problem 1. **[10 points]** 222z Calculate the number of ways to place 10 balls in four boxes with at least one in each box and at most four in each box.

First place one ball in each box. Then we must distribute six balls in four boxes in such a way that no more than three balls are in the same box.

We use inclusion/exclusion

Number of ways to place six balls in 4 boxes: $\binom{6+4-1}{6} = \binom{9}{6}$

Number of ways to place six balls in 4 boxes such that there are four balls in a box = number of ways to choose box with three balls * number of ways to place two balls in three boxes = $4 \cdot \binom{4}{2}$.

Number of ways to place six balls in 4 boxes such that there are five balls in a box = $4 \cdot \binom{3}{1}$.

Number of ways to place six balls in 4 boxes such that there are six balls in a box = 4.

So the total number of ways is

$\binom{9}{6} - 4\left(\binom{4}{2} + \binom{3}{1} + 1\right) = 44$

Problem 2. **[10 points]**

(a) Show that

$$\sum_{i=0}^{r} \binom{n+i}{i} = \binom{n+r+1}{n}$$

We will prove it by induction on r. Let $P(r) : \sum_{i=0}^{r} \binom{n+i}{i} = \binom{n+r+1}{r}$.
Base case: n = 0;
$1 = \binom{n}{0} = \binom{n+1}{0} = 1$
Inductive step:

$$\begin{aligned}\sum_{i=0}^{r+1}\binom{n+i}{i} &= \sum_{i=0}^{r}\binom{n+i}{i}+\binom{n+r+1}{r+1}\\ &= \binom{n+r+1}{r}+\binom{n+r+1}{r+1}\\ &= \binom{n+r+2}{r+1}\end{aligned}$$

where in the last step we used Pascal's identity.

(b) Find a formula for the number of ways a $2n+1$ game world series can be played, that terminates after one team wins $n+1$ games. The two teams are distinguishable. Use part (a) to write your answer in closed-form.

Fix a winning team, say team 1. The game can terminate in any day from the $(n+1)$st day to the $(2n+1)$st day. If the game terminates on the ith day, then there are $\binom{i-1}{n}$ ways to arrange the n victories of team 1 in the past $i-1$ days. Therefore the total number of ways that team 1 can win is

$$\sum_{i=n+1}^{2n+1}\binom{i-1}{n}$$

By symmetry, this is also the number of ways that team 2 can win, therefore the total number of ways the world series can occur is

$$2\sum_{i=n+1}^{2n+1}\binom{i-1}{n} = 2\sum_{i=0}^{n}\binom{n+i}{n} = 2\sum_{i=0}^{n}\binom{n+i}{i} == 2\binom{2n+1}{n}$$

Problem 3. **[10 points]**

(a) $3\cdot 3\cdot 3\cdot 3 = 81$.

(b) 14. We can put either all 4 balls in one urn, 3 in one, 2 in one and 2 in another, or 2 in one and 1 in each of the other two. The balls are distinguishable, so there are $\binom{4}{3}$ ways of putting 3 in one. There are $\binom{4}{2}$ ways of putting 2 in one and 1 in each of the other two. There are $\binom{4}{2}/2$ of putting 2 in one and 2 in another. So the total number of ways is

$$1+\binom{4}{3}+\frac{1}{2}\binom{4}{2}+\binom{4}{2} = 14.$$

(c) 15.
This is a stars and bars problem with the balls being stars and the urns bars. The answer is $\binom{4+3-1}{4} = \binom{6}{4} = 15$.

(d) 4. This is the number of ways to partition a multiset $\{1, 1, 1, 1\}$ of 4 indistinguishable objects into 3 (possibly empty) subsets. We can either put $\{1, 1, 1, 1, \}, \emptyset, \emptyset$, or $\{1, 1, 1\}, \{1\}, \emptyset$, or $\{1, 1\}, \{1, 1\}, \emptyset$, or $\{1, 1\}, \{1\}, \{1\}$. The ordering in which we list the subsets doesn't matter because the urns are indistinguishable.

Problem 4. [10 points]

(a) Design a huffman code for the following 8-letter alphabet with frequency counts as follows:

A .45
B .01
C .16
D .03
E .05
F .07
G .06
H .17

A: 0
B: 10000
C: 110
D: 10001
E: 1001
F: 1011
G: 1010
H: 111

(b) Give the code for the string "AHAACDEB".

01110011010001100110000

(c) How many bits are needed to encode a message with 100 letters, containing 45 A's, 1 B, 16 C's, 3 D's, 5 E's, 7 F's, 6 G's and 17 H's using this code?

We use the formula

$$length = \sum_{i=1}^{k} n_i length(i)$$

$$length = 45 \cdot 1 + 1 \cdot 5 + 16 \cdot 3 + 3 \cdot 5 + 5 \cdot 4 + 7 \cdot 4 + 6 \cdot 4 + 17 \cdot 3 = 236$$

Problem 5. [10 points]

How many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people that includes the bride and groom if

(a) The bride must be in the picture?

There are 6 ways to position the bride, $P(9,5)$ ways to choose and arrange the others. This gives a total of $\frac{6 \cdot 9!}{4!}$ ways.

(b) The bride and the groom must be in the picture?

There are 6 ways to place the bride and 5 to place the groom. There are $P(8,4)$ ways to choose and arrange the people. Total : $6 \cdot 5 \cdot \frac{8!}{4!}$ ways.

(c) Exactly one of the bride and groom must be in the picture?

2 ways to choose between bride or groom, 6 ways to place whichever one is chosen, and $P(8,5)$ to choose and arrange the remaining people. Total : $2 \cdot 8!$ ways.

Problem 6. Seven women and nine men are on the faculty of a school's math department. How many ways are there to select a committee of 5 members if

(a) at least 1 woman must be on the committee?

We need to count all possible combinations of people such that there is at least one woman in every combination, but we must remember not to count any combinations multiple times.

We can have committees with

1 woman, 4 men: $\binom{7}{1}\binom{9}{4}$

2 women, 3 men: $\binom{7}{2}\binom{9}{3}$

3 women, 2 men: $\binom{7}{3}\binom{9}{2}$

4 women, 1 man: $\binom{7}{4}\binom{9}{1}$

5 women, 0 men: $\binom{7}{5}\binom{9}{0}$

So there are $\binom{7}{1}\binom{9}{4}+\binom{7}{2}\binom{9}{3}+\binom{7}{3}\binom{9}{2}+\binom{7}{4}\binom{9}{1}+\binom{7}{5}\binom{9}{0} = 4242$ different possibilities for committees.

Another way to solve this problem is to say that there are $\binom{16}{5}$ different committees, and $\binom{9}{5}$ committees of just men. So there are $\binom{16}{5}-\binom{9}{5}$ different possibilities for committees.

Note that $7 \cdot \binom{15}{4}$ is not a correct answer. The product rule only applies when you are choosing two separate sets of items which are independent of each other. In this case, a woman is chosen and then the remainder of the committee is chosen, but since there may be women in the last set of 4 chosen, we will be double counting. For example, if A and B are women, and C, D, and E are men, then we will count $A, \{BCDE\}$ as well as $B, \{ACDE\}$.

(b) at least 1 man and 1 woman must be on the committee? We can have committees with

1 woman, 4 men: $\binom{7}{1}\binom{9}{4}$
2 women, 3 men: $\binom{7}{2}\binom{9}{3}$
3 women, 2 men: $\binom{7}{3}\binom{9}{2}$
4 women, 1 man: $\binom{7}{4}\binom{9}{1}$

So there are $\binom{7}{1}\binom{9}{4} + \binom{7}{2}\binom{9}{3} + \binom{7}{3}\binom{9}{2} + \binom{7}{4}\binom{9}{1} = 4221$ different possibilities for committees.

Again, this is also equal to the number of different committees, minus the number of committees with no men, and minus the number of committees with no women: $\binom{16}{5} - \binom{9}{5} - \binom{7}{5}$

(c) there must be more men than women on the committee?

For there to be more men than women, we must have 3 or more men on the committee.

5 men, 0 women: $\binom{9}{5}\binom{7}{0}$
4 men, 1 woman $\binom{9}{4}\binom{7}{1}$
3 men, 2 women: $\binom{9}{3}\binom{7}{2}$

$\binom{9}{5}\binom{7}{0} + \binom{9}{4}\binom{7}{1} + \binom{9}{3}\binom{7}{2} = 2772$

Problem 7. **(a)** Describe a method for guessing 2 cards when 7 out of 9 are revealed.

There must be $\lceil 9/4 \rceil = 3$ cards with the same color, so we choose to hide two of them. The first card revealed fixes the color and the offset. There are six cards to code the position. These suffice. For example, the first three cards can fix the position relative to the offset fixed by the first card, and the other three card to describe the position relative to the position of the first hidden card. Note that the hidden card must be chosen so that both offsets are less than or equal to 6.

(b) Prove that you cannot guess 2 cards when 4 out of 6 are revealed.

There are $\binom{52}{6}$ possible six-card hands. There are $P(52, 4) = \frac{52!}{48!}$ four-card hands that can be displayed (with order counted), therefore by pigeonhole there are at least $\left\lceil \frac{\binom{52}{6}}{\frac{52!}{48!}} \right\rceil = 4$ six-card hands result in the same four-card display. Therefore, the magician cannot correctly guess in this case since these 4 hands must have different hidden cards.

Problem 8. **[10 points]**

A shelf holds 12 books in a row. How many ways are there to choose five books so that no two adjacent books are chosen?

There are 5 bars (chosen books), and therefore there are 6 places where the 7 stars (non-chosen books) can fit (before the first bar, between the first and second bars,..., after the fifth bar). Each of the second through fifth of these slots must have at least one star in it, so that adjacent books are not chosen. Once we have placed these 4 stars, there are 3 stars left to be placed in 6 slots. The number of ways to do this is therefore $\binom{6+3-1}{3} = \binom{8}{3} = 56$.

Problem 9. **[10 points]**

Compute the (ordinary) generating function for each of the following sequences.

(a) $1/(1-x)$.

(b) $1 + x + x^2$.

(c) $x^2/(1-x)$.

(d) $x^3/(1-x)$.

(e) $x^3 + x^4 + 2x^5 + x^6/(1-x) = \frac{x^3}{1-x} + x^5$.

(f) $\sum_n x^{3n} = 1/(1-x^3)$.

(g) $(1+x)/(1-x)^3$.

(h) $\sum_n n(n+2)x^n = \sum_n n^2 x^n + 2\sum_n nx^n = \frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3} + 2\frac{x}{(1-x)^2} = \frac{2x^2}{(1-x)^3} + \frac{3x}{(1-x)^2}$

(i) $1/(1-3x) + 1/(1-5x) + 1/(1-7x)$.

(j) e^x.

Problem 10. **[10 points]**

(a) GF for $\langle a_k \rangle = \frac{x}{1-x^2} \implies A(x) = (\frac{x}{1-x^2})^4$.

(b) GF for $\langle a_k \rangle = \frac{1}{1-x^3} \implies A(x) = (\frac{1}{1-x^3})^4$.

(c) GF for: $\langle a_1 \rangle = 1,\ \langle a_2 \rangle = 1 + x,\ \langle a_3 \rangle, \langle a_4 \rangle = \frac{1}{1-x} \implies A(x) = \frac{1+x}{(1-x)^2}$.

(d) GF for: $\langle a_1 \rangle = x + x^3 + x^{11},\ \langle a_2 \rangle = x^2 + x^4 + x^5,\ \langle a_3 \rangle, \langle a_4 \rangle = \frac{1}{1-x} \implies$ $A(x) = \frac{(x+x^3+x^{11})(x^2+x^4+x^5)}{(1-x)^2}$.