Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professors Tom Leighton and Santosh Vempala

Handout 26
November 4, 1997

Problem Set 9

Due: November 18, 1997.

Reading: 4.4-4.5 of Rosen

Problem 1. [5 points]

Consider playing the *Lets Make a Deal*, where there are three prizes: a "Gift Certificate" from Toscanini, a "Nerd Pocket Protector", and a "Nerd Pride Button". Your goal is to win the gift certificate.

(a) Suppose you (the contestant) happen to know that Carol always reveals the "Nerd Pocket Protector", if you did not choose the box covering it.

Does this additional information change the odds of winning for you? Is there a better strategy? By how much (if any) does this information increase your chances of winning?

(b) You are playing according to the original assumptions, but before you make your initial choice, you happen to see Carol placing a "Nerd Pride Button" underneath Box 1. What should you do to try to win the prize (the ice cream)? What is the probability that you win?

Problem 2. [10 points]

Consider a three by three array of squares as in the game of tic-tac-to.

(a) If three squares are picked at random (independently and with the same probability), what is the probability that these squares will form a line? (horizontal, vertical, or diagonal)

(b) If you pick four squares at random, what is the probability that three of them will be on a line?

Problem 3. [10 points]

(a) You roll two dice. What is the probability that their sum is not 6?

(b) You roll three dice. What is the probability that that any of the *pairwise* sums is not 6?

Problem 4. **[10 points]**

(a) Prove that $Pr(\overline{A}|C) = 1 - Pr(A|C)$ for any events A and C.

(b) Is it possible for $Pr(A|B)$ and $Pr(A|C)$ to be very small, if $Pr(A|B \cap C)$ is large? Give a counterexample or a proof to justify your answer.

Problem 5. **[5 points]**

Three Prisoners A,B, and C have applied for parole. The parole board has decided to randomly choose two of the three to be released, and the prisoners know this but not which two. A warder friend of prisoner A knows who are to be released. A realizes that it would be unethical to ask if he is going to be released, but thinks of asking for the name of *one prisoner other than himself* who is to be released. Before he asks his chances are 2/3, but he thinks that if he asks and the warder says "B will be released", his own chances will become 1/2 since either A or C will be the only remaining candidates. So A decides not to reduce his chances by asking. Is A's argument correct? Show what his chances are after asking. You may assume that if the warder has a choice of which name to reveal, then he picks one at random.

Problem 6. **[10 points]**

We know from statistics that when a policeman gets shot he/she survives with probability 0.8 if he/she wears a bullet-proof vest and survives with probability 0.12 if the policeman doesn't wear a vest. Given that 15% of the policemen wear bullet-proof vests, what is the probability that a random policeman was wearing a bullet-proof vest if he/she survived a firearm attack?

Problem 7. **[10 points]**

A, B, and C are to fight a three-cornered pistol duel. They all know that A's chance of hitting the target is 0.3, C's is 0.5 and B never misses. They are to shoot in the sequence A, B, C and fire at their choice of target until one person is left standing. Show that A is better off missing deliberately the first shot than attempting to hit either B or C. Assume that every player plays optimally.

(*Hint: consider the situation where B is dead and C shoots next. What is the probability P that A ultimately survives? Show that $P = .5(.3 + .7P)$.*)

Problem 8. **[10 points]**

(a) Show that if two events A and B are independent, then the events A and $\overline{B}$ are also independent.

(b) Two numbers x and y are selected at random from the set $\{1, \ldots, 100\}$. Let the events A, B and C be defined as follows: $A = \{x > 50\}$, $B = \{y > 50\}$ and $C = \{x > y\}$. Are the events A and B independent? Are A and C independent?

Problem 9. **[10 points]**

You are given a coin of bias p (prob of heads = p; tails = 1-p, $0 < p < 1$). The value of p is not known and you are not allowed to guess by sampling.

Using this, how can you get a coin of bias:

1. p^2
2. $\frac{1}{2}p^2 + \frac{1}{4}p$
3. $\frac{1}{2} - \frac{1}{4}p$

Problem 10. **[10 points]**

Consider three loaded dice such that

Die A always comes up 3
Die B is 2 with prob p and 5 otherwise
Die C is 4 with prob p and 1 otherwise.

Find a value for p such that A beats B, B beats A, and C beats A all with probability p.

Problem 11. **[10 points]**

In a complex system with 100,000 different components. The system contains 10 critical components, each failing with probability 1/1000 and 100 secondary components each failing with probability 1/100. It is known that secondary failures are pairwise independent. The system crashes if one critical component or two secondary components fail.

(a) Determine an upper bound on the probability that the system will crash.

(b) Determine a lower bound on the probability that the system will crash.

(c) Where is the biggest weakness? In the critical, or in the secondary components?