# Quiz 1 Solutions

**Problem 1. 10 points**

Use induction to prove that $\forall n \geq 1$,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

*Proof by induction.*
Let $P(n)$ be "$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$."

**Base case:** $n = 1$. $P(1)$ is true since $\frac{1}{1\cdot 2} = \frac{1}{(1+1)}$.

**Inductive step:** Assume $P(n)$ is true, in order to show that $P(n+1)$ is true.

$$\begin{aligned}
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + ... + \frac{1}{(n+1)\cdot((n+1)+1)} &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + ... \frac{1}{n(n+1)} + \frac{1}{(n+1)((n+1)+1)} \\
&= \frac{n}{n+1} + \frac{1}{(n+1)\cdot((n+1)+1)} \text{ true by } \textbf{I.H.} \\
&= \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n^2+2n+1}{(n+1)(n+2)} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} \\
&= \frac{n+1}{n+2} \\
&= \frac{n+1}{(n+1)+1}
\end{aligned}$$

**Problem 2. 10 points**

**(a)** **[6 points]** Cast each of the following statements in propositional symbol form.

Definitions:

1. $C$ = set of classes.
2. $M$ = set of majors.
3. for $x \in C$, $H(x)$ means $x$ is hard.
4. for $x \in C, y \in M$, $R(x, y)$ means $x$ is required for $y$.

Statements:

1. **[2 points]** At least one class is hard.

   $\exists x \in C : H(x)$

2. **[2 points]** Every major requires at least one hard class.

   $\forall y \in M \exists x \in C : H(x) \wedge R(x, y)$

3. **[2 points]** No class is required for two or more majors.

   $\neg \exists x \in C, \exists y_1 \in M \exists y_2 \in M : (y_1 \neq y_2) \wedge R(x, y_1) \wedge R(x, y_2)$

**(b)** **[4 points]** Write the negation of $\forall y \exists x H(x) \wedge R(x, y)$ in terms of $\neg H(x)$ and $\neg R(x, y)$ (and without using any other negation symbols – i.e., $\neg\forall$ is not allowed.)

$\exists y \forall x \neg H(x) \vee \neg R(x, y)$

**Problem 3. [10 points]**

What denominations of postage can be made with an unlimited supply of stamps of $5, 6$, and 7 cents? Be sure to prove your answer.

The set of possible postage denominations is $S = \{5, 6, 7\} \cup \{n \in \mathbb{N}; n \geq 10\}$

The cases for $5, 6, 7$ are immediate. Also, it is clear that $1, 2, 3, 4, 8, 9$ cannot be made.

We use induction for the rest.

Let $P(n) =$ " $n$ can be made with stamps of value 5,6,and 7 cents".

**Claim:** $\forall n \geq 10 : P(n)$

**Base case:** $n = 10$ can be made with two 5-cent stamps

**Inductive hypothesis:** Assume $P(n)$ is true in order to show that $P(n+1)$ is true.

We want to show that a postage of $n + 1$ can be made by the available stamps. By the inductive hypothesis $n$ can be made by the available stamps. If $n$ can be made with a 5, we take out the 5 cent stamp and replace it with a 6 cent stamp. If $n$ can be made with a 6, we take out the 6 cent stamp and replace it with a 7 cent stamp. If $n$ contains neither a 5 or a 6, then $n$ must contain two 7 cent stamps (because it's bigger than 10). So we can replace the two 7 cent stamps with three 5 cent stamps. The induction is complete.

Here's an alternative proof that uses strong induction. We provide the inductive step.

Assume $P(10) \wedge P(11) \wedge \ldots \wedge P(n)$, in order to show $P(n+1)$.

Note that $P(n-4) \Rightarrow P(n+1)$ since we can add a 5-cent stamp to $n-4$. So the inductive hypothesis is proved for $n-4 \geq 10 \Rightarrow n \geq 14$. Hence we must check the inductive hypothesis for $10 \leq n < 14$.

10 is just the base case.
$11 = 5 + 6$
$12 = 6 + 6$
$13 = 6 + 7$
$14 = 7 + 7$

The induction is complete.

**Problem 4. [20 points]**

Recall that a coloring of a graph is an assignment of a color to each vertex. A graph is defined to be $k$-colorable if it has a coloring using at most $k$ colors such that no two adjacent vertices have the same color. (*Note: we consider only graphs without loops in this problem*)

**(a) [5 points]**

Prove that if a graph has maximum node degree $k$ then it is $(k+1)$-colorable.

Proof by induction on the number of vertices:

Inductive hypothesis:
$P(n) =$"Every graph of n vertices with maximum node degree $k$ is $(k+1)$ colorable."

**Base case:** $(n = 1)$
The only graph with a single vertex can always be colored with one color, so for any $k \geq 0$, it is $(k+1)$-colorable.

**Inductive step:**

Assume $P(n)$ in order to prove $P(n+1)$.
Given any graph $G_{n+1}$ with n+1 vertices and maximum degree $k$, remove one vertex, $v$, to produce a graph $G_n$ with $n$ vertices. Since we have not added any edges, $G_n$ must have maximum vertex degree at most $k$. By the inductive hypothesis, we can color $G_n$ with $k+1$ colors. We can then add $v$ back into the graph. Since $v$ has at most $k$ neighbors, we can color it with one of the original $k+1$ colors, giving a $(k+1)$-coloring for $G_{n+1}$.

**(b) [15 points]**

A graph is defined to be regular if every vertex in the graph has the same degree. Show that if a connected graph has maximum node degree $k$ and is not regular then it is $k$-colorable. (*You may use the result from part (a) without proof*).

Proof by strong induction on the number of vertices:

Inductive hypothesis:

$P(n) =$"Every connected graph of $n$ vertices and maximum node degree $k$ that is not regular is $k$-colorable."

**Base case:** (n=1)

The graph with one vertex is always regular, so $P(1)$ holds vacuously.

**Inductive step:**

Assume $P(1) \wedge P(2) \wedge \cdots \wedge P(n)$, in order to prove $P(n+1)$.

Given any graph $G_{n+1}$ with $n+1$ vertices and maximum vertex degree $k$, $P(n+1)$ holds trivially if the graph is not connected or is regular. Therefore assume $G_{n+1}$ is connected and not regular. Then, since $G_{n+1}$ is not regular, it must have some vertex of degree less than $k$. Remove that vertex, $v$, to produce a graph $G_n$. Then we can consider each connected component $C_i$ of $G_n$:

Case 1: $C_i$ is regular. Since $G_{n+1}$ was connected, $v$ must have been connected to at least one node in $C_i$. Therefore, at least one node of $C_i$ must have a lower degree than it did before $v$ was removed. Therefore $C_i$ has at least one node of degree less than $k$. Since $C_i$ is regular, all nodes must therefore have degree less than $k$, meaning $C_i$ has a maximum node degree less than $k$. Therefore $C_i$ is $k$-colorable by part (a).

Case 2: $C_i$ is not regular. $C_i$ must have degree at most $k$ and it is connected by definition, so by the inductive hypothesis, it is $k$-colorable.

Since all components of $G_n$ are $k$-colorable, $G_n$ is $k$-colorable. We can therefore assume $G_n$ is assigned some $k$-coloring. Now we can reattach $v$. Since $v$ has less than $k$ neighbors, there must be some $j \leq k$ such that $v$ is not adjacent to any node of color $j$. Thus, we can color $v$ with color $j$, producing a $k$-coloring of $G_{n+1}$. Therefore, if $G_{n+1}$ has maximum vertex degree $k$ and is not regular then it is $k$-colorable.

**Problem 5. [10 points]**

Luke finished his lightsaber practice, and sits down for a game of 4 dimensional chess. The space of moves is a four dimensional $11 \times 11 \times 11 \times 11$ grid with cells $\{(x, y, z, w) | -5 \leq x, y, z, w \leq 5; x, y, z, w \in \mathbb{Z}\}$. If his king moves 1 unit in each of two different directions at each step, and he starts at $(1, 0, 0, 0)$, can his king reach $(5, 4, -1, 2)$? Express the possible moves as a state machine and find an invariant to prove your answer.

The state is the possible positions $(x, y, z, w)$ on the chess-board $-5 \leq x, y, z, w \leq 5; x, y, z, w \in \mathbb{Z}$. The transitions are the movements of the king, i.e., exactly two coordinates change by $\pm 1$ in a way such that the king remains in the chess board. The initial state is $(1, 0, 0, 0)$.

The invariant is $P(x, y, z, w) : x + y + z + w$ is odd

The invariant is certainly true for the initial state, $(1, 0, 0, 0)$.

Assume that it is true at state $(x, y, z, w)$. At the next state transition to state $(x', y', z', w')$ two of the state variables are changed by $\pm 1$, therefore the sum of the invariant is changed by either $-2, 0$, or $2$. Therefore $(x' + y' + z' + w')$ remains odd. The invariant is proved.

The state $(5, 4, -1, 2)$ does not satisfy the invariant, therefore it is not a reachable state.

**Problem 6. [9 points]**

For each of the following pairs of functions $f(n)$ and $g(n)$, fill in the blank with $o$, $O$, $\omega$, $\Omega$, and $\Theta$ depending on whether $f(n) = o(g(n))$, $f(n) = O(g(n))$, $f(n) = \omega(g(n))$, $f(n) = \Omega(g(n))$, or $f(n) = \Theta(g(n))$. If there is more than one relation between $f(n)$ and $g(n)$, write the strongest one. You do not need to supply any proofs.

$$n^2 = o(n^3)$$

$$2^n = o(3^n)$$

$$\log_2 n = \Theta(\log_3 n)$$

$$n^3 = o(2^n)$$

$$\frac{n}{\log n} = o(n)$$

$$\log_2 n = o(\sqrt{n})$$

$$8^{\log_2 n} = \Theta(n^{\log_2 8})$$

$$(n!)^2 = \omega(n^n)$$

$$2^{\log_3 n} = o(n^2)$$

**Problem 7. [10 points]**

Your answer in the following problems should not contain $\sum$ or $\prod$ and should be in closed form.

**(a) [4 points]**

Find tight asymptotic ($\Theta$) bounds for

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

We use the integration method for bounding summations.

$$\begin{aligned}
\int_1^n \frac{1}{\sqrt{x}}dx &\leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} &\leq 1 + \int_1^n \frac{1}{\sqrt{x}}dx \\
2x^{\frac{1}{2}}|_1^n &\leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} &\leq 1 + 2x^{\frac{1}{2}}|_1^n \\
2\sqrt{n} - 2 &\leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} &\leq 2\sqrt{n} - 1
\end{aligned}$$

That is

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} = \Theta(\sqrt{n}).$$

**(b) [6 points]**

Evaluate the following product

$$\prod_{i=1}^{\infty} e^{(2^{-i})}$$

$$\begin{aligned}
\prod_{i=1}^{\infty} e^{(2^{-i})} &= e^{\sum_{i=1}^{\infty} 2^{-i}} \\
&= e^1 \\
&= e
\end{aligned}$$

**Problem 8.** **[11 points]**

Find tight asymptotic bounds for the following recurrences. For each recurrence $T(0) = 1$ and $n \in \mathbb{N}$.

**(a) [3 points]**
$T(n) = 2T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n$

We use Akra-Bazzi.
$h_1(n) = \left\lfloor \frac{n}{4} \right\rfloor - \frac{n}{4}$, $h_2(n) = \left\lfloor \frac{n}{3} \right\rfloor - \frac{n}{3}$. $g(n) = n$. $a_1 = 2, a_2 = 1$. $b_1 = \frac{1}{4}, b_2 = \frac{1}{3}$.
Setting $p = 1$ we get $2\frac{1}{4} + \frac{1}{3} = \frac{5}{6} < 1$. We conclude that $p < 1$, therefore the recurrence is of the order of the inhomogeneous part $g(n) = n$. Therefore $T(n) = \Theta(n)$.

**(b) [3 points]**
$T(n) = 4T\left(\left\lfloor \frac{n}{2} + \sqrt{n} \right\rfloor\right) + n^2$

We use Akra-Bazzi.
$h(n) = \left\lfloor \frac{n}{2} + \sqrt{n} \right\rfloor - \frac{n}{2}$. $a_1 = 4, b_1 = \frac{1}{2}, g(n) = n^2$
$4\left(\frac{1}{2}\right)^2 = 1 \Rightarrow p = 2$

$$T(n) = \Theta(n^2(1 + \int_1^n \frac{u^2}{u^{2+1}} du)) = \Theta(n^2(1 + \int_1^n u^{-1} du)) = \Theta(n^2 \log n).$$

**(c) [5 points]**

A society of devil-worshipers meets every week in a catacomb to initiate new members. Members who have been in the society for two or more weeks initiate four new members each and members who have been in the society for only one week initiate one new member each. On week 0 there is one devil-worshiper. There are two devil-worshipers on week 1.

Write a recurrence relation for the number of members $D(n)$ in the society on the $n$th week.

**You do NOT need to solve the recurrence. Be sure to include the base cases.**

The number of devil-worshipers on week $n$ is the number of devil-worshipers that were around 2 weeks before plus their new recruits, so $5D(n-2)$, and the number of devil worshipers that were around only last week, $D(n-2) - D(n-1)$, plus their new recruits. So the total number of devil-worshipers on the $n$th week is

$D(n) = 5D(n-2) + 2(D(n-1) - D(n-2)) = 3D(n-2) + 2D(n-1)$

The initial conditions are

$D(0) = 1$

$D(1) = 2$

**Problem 9.** **[10 points]**

Suppose that you have $n$ balls where $n-1$ of them have the same weight and one of them is heavier. You also have a scale that can compare any two groups of balls and tell you which group is heavier (or if they are equal).

**(a)** **[4 points]**
Describe an algorithm to find the heavier ball when $n = 9$ using two weighings.

Weigh balls 1-3 against balls 4-6.
If they are equal, weigh 7 against 8. If they are unequal, choose the heavier. If they are equal choose 9.

If 1-3 is heavier, weigh 1 against 2. If they are unequal, choose the heavier. If they are equal choose 3.
If 4-6 is heavier, weigh 4 against 5. If they are unequal, choose the heavier. If they are equal choose 6.

**(b)** **[6 points]**
Prove that there is no algorithm that uses two weighings for $n = 10$.

There are 10 possibilities for the heavy ball. With 2 weighings, there are $3^2 = 9$ possible outcomes of the algorithm. Since $10 > 9$, the pigeonhole princeple means that some outcomes of the weighings cannot distinguish between two of the possibilities.